6.5. Hipparchus, Menelaus, Ptolemy, and Greek Trigonometry

Note. In this section, we consider the beginnings of Greek trigonometry, mostly as it grew out of measurements of the sky and the beginnings of mathematical astronomy. In addition to the three people mentioned in the title of the section, we consider Theodosius.

Note 6.5.A. Theodosius of Bithynia (circa 160 BCE–circa 90 BCE) was a Greek mathematician and astronomer born in Bithynia, Anatolia (in modern day northern Turkey).

An (imagined) image of Theodosius from the
Ancient Greece Reloaded website (accessed 9/1/2023)

The sky appears as a sphere around the Earth on which the stars and planets move (the "celestial sphere"), so the study of the geometry of the sphere is an early part of the study of astronomy. The subject was developed before Euclid (circa 325

BCE–circa 265 BCE) and there were books in existence on great and small circles on the sphere, from which Euclid (and others) quoted. The propositions on this topic, along with other propositions of a purely astronomical interest (as opposed to a more general geometric interest) were collected in three books by Thodosius titled *Sphaerica* (or *Spherics*). This work survives and Johan Ludvig Heiberg (see Note 5.3.J of Section 5.3. Euclid's "Elements") published *Theodosius. Sphaerica, Greek and Latin Text* (Berlin: Weidmannsche Buchhandlung, 1927). There is also a French translation: Paul ver Ecke, *Theodosius. Les sphériques de Theodose de Tripoli*, translation with introduction and notes (Paris: Blanchard, 1959). However, there does not seem to be an English version of *Sphaerica*. There are two nice descriptions in English of the contents, with illustrations. Both are by R.S.D. Thomas. One is "The Definitions and Theorems of the Spherics of Theodosios," In: Zack, M., Schlimm, D. (eds.) *Research in History and Philosophy of Mathematics*, Proceedings of the Canadian Society for History and Philosophy of Mathematics/Société canadienne d'histoire et de philosophie des mat'ematiques. Birkhäuser, Cham (2018). A version of this paper is online at the Waseda University Homepage of Nathan Camillo Sidoli. Another work of R.S.D. Thomas, concerning the contents of Book I, is "An Appreciation of the First Book of Spherics," *Mathematics Magazine*, **91**(1), 3–15 (February 2018). This is not readily available online, but you can access it through the ETSU Sherrod Library catalog using JSTOR. Euclid's *Elements* largely exclude the geometry of the sphere, probably because such results are viewed as part of astronomy rather than of geometry. The beginning Theodosius' *Sphaerica* includes these definitions (quoting from Thomas' "An Appreciation of the First Book of Spherics"):

Definition 1. A *sphere* is a solid figure contained by a single surface, all lines to which, falling from a single point that lies within the figure, are equal to one another.

Definition 2. *Center of the sphere* is the point.

Definition 5. *Pole of a circle in a sphere* names a point on the surface of the sphere all lines from which, falling on the circumference of the circle, are equal to one another.

In Proposition 1 it is shown that any plane section (i.e., the intersection of a plane and) of a sphere is a circle. Such a plane intersection can be used to find a diameter of the sphere and then to find the center of the sphere (Proposition 2). Proposition 20 concerns the construction of the great circle through any two given points (not opposite each other on the sphere) on the sphere. Proposition 21 gives a way to find the pole of any given circular section. There are 22 propositions in Book I. Book II begins with a definition of circles on a sphere which touch one another (i.e., to tangent circles). In Propositions 1 and 2 of Book II, parallel circular sections are shown to have the same poles, and the converse of this. Proposition 9 shows that if two circles on a sphere cut one another, then the great circle drawn through their poles bisects the intercepted segments of the circles. Book III contains propositions of purely astronomical interest, but stated as propositions in pure geometry. Theodosius' results in *Sphaerica* concern comparing certain arcs and determining which is greater. However, actual numerical values are not associated with the arcs, making them useless for astronomical measurements (such as describing the location of a star in the night sky at different times). In order to quantify these quantities, trigonometry is needed. *Sphaerica* contains no trigonometry (though it is a "prelude") to trigonometry. In addition to *Sphaerica*, Theodosius wrote *On Habitations* and *On Days and Nights*, which have survived in Greek. In *On Habitations* there are 12 propositions related to the different observed phenomena due to the daily rotation of the Earth (or, as it was more likely interpreted, the daily rotation of the celestial sphere). In *On Days and Nights*, there are 32 propositions (over two books) concerning the movement of the sun around the ecliptic (a great circle on the celestial sphere that is at an angle of 23.4° to the celestial equator, reflecting the tilt of the Earth's axis by 23.4° to the plane of the solar system), the solstices, and the equinoxes. Theodosius is also thought to have written a commentary on Archimedes' *Method*, but this commentary does not survive. It seems that the work of Theodorius does not contain new results, but instead his works are compilations of previously known results. The source for this note is Thomas Heath's *A History of Greek Mathematics, Volume 2* (Oxford: Clarendon Press, 1921), pages 245–252.

Note 6.5.B. Hipparchus of Rhodes (also "Hipparchus of Nicaea"; 190 BCE–120 BCE) is known to have made astronomical observations from Rhodes between 161 BCE and 126 BCE (according to Ptolemy [circa 85 CE–165 CE). He cataloged the positions of over 850 stars (using only naked eye observations…telescopes were not invented for another 1700 years). His precise measurements of the location of these stars in the night sky inspired the European Space Agency to name a satellite it launched in 1989 "Hipparcos." The satellite was designed to make very precise measurements of positions, proper motions, and parallaxes of stars. The name "Hipparcos" is an acronym for **HI**gh **P**recision **PAR**allax **C**Ollecting **S**atellite (see the figure below, right). Sadly, very little of his work survives, so we only know him from references by others.


Image of Hipparchus (left) form the the MacTutor biography webpage of Hipparchus, and image of European Space Agency satellite *Hipparcos* (right) from the Wikipedia webpage on Hipparcos (both accessed 9/1/2023).

According to Thomas Heath, "[Hipparchus'] greatest [accomplishment] is perhaps his discovery of the precession of the equinoxes" (page 254 of *A History of Greek Mathematics, Volume 2*). By measuring the location of the bright star Spica in Virgo relative to the location of the point of the autumnal equinox, and comparing this to observations made 154 years earlier by Timocharis (circa 320 BCE–260 BCE). Based on this, he calculated that the angle between Spica and the point of the autumnal equinox had changed by around 2° during that time. We know today that this is the result of the "wobble" (or "precession") of the Earth's axis of rotation. As a result of this, the coordinates used by astronomers to describe locations in the sky have to be periodically revised. In his work *On the Length of the Year*, Hipparchus compared observations he made of the summer solstice to those made by Aristarchus 145 years earlier, to deduce that the length of the day is $1/300^{\text{th}}$ of a day shorter than the accepted value at the time of $365\,1/4$ days. The Gregorian calendar of 1582 is based on a year-length of 365.2425 which is $1/0.0075^{\text{th}} \approx 1/133^{\text{th}}$ of a day shorter than $365\,1/4$ days. Hipparchus similarly estimated the mean lunar month as 29 days 12 hours 44 minutes $2\,1/2$ seconds in length, which is with one second of the currently accepted value. Aristarchus of Samos (circa 310 BCE–circa 230 BCE), who proposed a heliocentric solar system almost 1800 years before Copernicus, estimated the ratio of the distance d_S from the Earth to the Sun, to the distance d_M from the Earth to the Moon is is between 18 and 20, $18 \le d_S/d_M \le 20$ (see Problem Study 6.1(a)). Hipparchus improved this estimate based on the their apparent diameters and changes in them. With D as the diameter of the Earth, he estimated the Sun was at a mean distance of $1245D$ and the moon at a mean distance of $33\frac{2}{3}D$, so that the Hipparchus estimates the ratio of the distances as $d_S/d_M = 1245D/(33\frac{2}{3}D) \approx 40$. The average distance from the Earth to the Sun is today known to be 93,803,000 miles, and he average distance from the Earth to the Moon is 238,855 miles. So an accurate value of the ratio is $d_S/d_M = 93,803,000/238,855 \approx 393$ (less impressive than Hipparchus' estimate of the lunar month, eh?). Hipparchus was an advocate of using latitude and longitude to measure locations on the Earth (similar to the coordinates used to measure locations on the celestial sphere). The source for this note is Thomas Heath's *A History of Greek Mathematics, Volume 2* (Oxford: Clarendon Press, 1921), pages 253–256.

Note 6.5.C. We now address Hipparchus' contribution to trigonometry. Hipparchus is the earliest one for whom there is documentary evidence of the systematic use of trigonometry. The evidence is given by Theon of Alexandria (circa 335 CE–circa 405 CE) says that Hipparcus wrote a treatise in twelve books on straight lines (i.e., chords) in a circle (and also that Menelaus [circa 70 CE–circa 130 CE] wrote a similar work in six books). Also, Pappus of Alexandria (circa 290 CE–circa 350 CE) observes that in his book *On the Rising of the Twelve Signs of the Zodiac*, Hipparchus showed that equal arcs of the zodiac have setting times in a certain that varies with location (i.e., latitude). With this result, Hipparchus has actually quantified amounts of time, unlike his predecessors such as Theodosius (see Note 6.5.A above). In Hipparchus' only surviving work, *Commentary on the Phaenomena of Eudoxus and Aratus*, he considers numerical lengths of arcs that stars trace out in the sky as seen from certain latitudes. In the twelve book treatise on chords mentioned by Theon included a table of chords. Hipparchus' tables of chords are known from the work of Ptolemy (circa 85 CE–circa 165 CE), which we describe below. A circle is considered and the radius of the circle is divided into 60 equal parts (so we take the radius to be 60 and the diameter to be 120). Ptolemy, believed to be adopting work of Hipparchus' (Eves, page 175) considers central angles of the circle at halg-degree intervals from 1/2° to 180°.

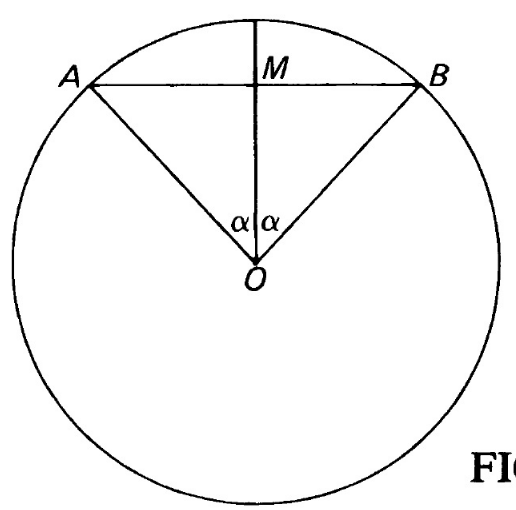


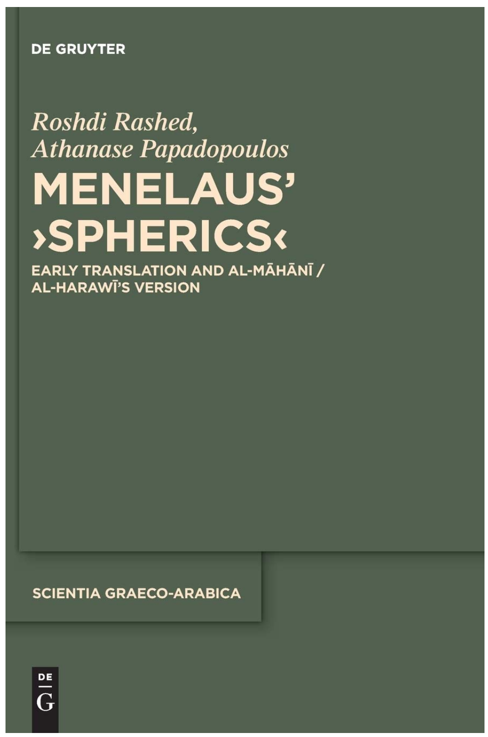
FIGURE 48

Base on Figure 48 above, we desire a relationship between the length of the chord AB and the central angle α. We define the chord function that gives the length of AB in terms of angle 2α, $\text{crd}(2\alpha)$, as

$$\sin\alpha = \frac{AM}{OA} = \frac{AB}{\text{diameter of circle}} = \frac{\text{crd}(2\alpha)}{120}.$$

What Ptolemy gives in his table is then (effectively) a table of the sine function as the angle varies from 0° to 90° in $1/4^\circ = 15'$ increments. In term of the work of Ptolemy, this is also discussed in my online notes for the history component of Introduction to Modern Geometry (MATH 4157/5157) on Section 5.1. Ptolemy and the Chord Function. The source for this note is Thomas Heath's *A History of Greek Mathematics, Volume 2* (Oxford: Clarendon Press, 1921), pages 257–260.

Note 6.5.D. We stated above in Note 6.5.C that Theon of Alexandria (circa 335 CE–circa 405 CE) mentions that Menelaus (circa 70 CE–circa 130 CE) wrote a work on chords in a circle in six books, titled *Chords in a Circle*. Pappus (circa 290 CE–circa 350 CE) in Book VI of his *Collection* says that Menelaus wrote a treatise on the rising and setting times of different arcs of the zodiac. Arabian records indicate three other works of Menelaus. First, he wrote *Elements of Geometry* in three books, which was edited by Thābit ibn Qurra (836 CE–February 18, 901 CE); Qurra is mentioned in the supplement Euclid's *Elements*—A 2,500 Year History as a translator of works by Euclid, Archimedes, and Apollonius. Second, he wrote a book on triangles. Third, he wrote a work with the title "On the Knowledge of the Discrete Quantity of Mixed Bodies," thought to be a book on hydrostatics. The book of main concern to us is his *Sphaerica*. This is in three books and is preserved in the Arabic (in several versions which differ if form but give a good idea of the specific content). A well-known Latin translation from Arabic is due to Gherard of Cremona (or "Gerard"; 1114–1187), who we will see again as a translator in Section 8.2. The Period of Transmission; see Note 8.2.C. In fact, A modern English version is still in print: Roshdi Rashed and Athanase Papadopoulos, *Menelaus' 'Spherics': Early Translation and al-Māhāni/al-Harawī's Version*, Scientia Graeco-Arabica 21 (Berlin; Boston: De Gruyter, 2017).

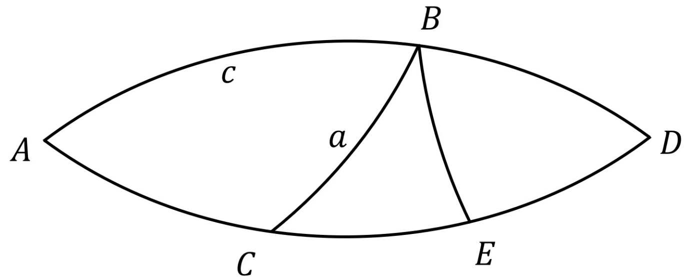


Note 6.5.E. Book I of *Sphaerica* gives for the first time the definition of a spherical triangle in his Definition 1. Menelaus states it in terms of the area included by arcs of great circles on the surface of a sphere (under the convention that each of the sides of the triangle is an arc less than a semicircle). The angles of a spherical triangle are the angles contained by the arcs of great circles on the sphere

(Definition 3). We consider here the usual notation of a, b, c for the sides of a spherical triangle and A, B, C for the opposite angles, respectively. We measure angles in degrees and, surprisingly perhaps, we measure the lengths of sides in degrees as well. This is reasonable since we will only compare lengths of sides and a side is determined by a central angle in the sphere. Comparing lengths of sides is then equivalent to comparing central angles, since the length equals (circumference)(central angle)/360°. Book I mostly concerns proving results for spherical triangles which correspond to Euclid's propositions for plane triangles. Some exceptions are Euclid's Propositions I.16 ("In any triangle, if one of the sides is produced, then the exterior angle is greater than either of the interior and opposite angles.") and I.32 ("In any triangle, if one of the sides is produced, then the exterior angle equals the sum of the two interior and opposite angles, and the sum of the three interior angles of the triangle equals two right angles [i.e., 180°]."), which do not hold for spherical triangles. Euclid's Proposition I.16 is replaced with Proposition I.10 of *Sphaerica*, which states:

Proposition I.10. For spherical triangle ABC, the exterior angle at C (i.e., angle $180° - C$) is less than or equal to A (i.e., $\angle A \geq 180° - C$) if and only if $c + a \geq 180°$, and exterior angle at C (i.e., angle $180° - C$) is greater than A (i.e., $\angle A < 180° - C$) if and only if $c + a < 180°$.

Proof. Consider triangle ABC in the figure below (where the edges are represented by arcs of a circle). Let D be the pole opposite to A (so that $AD = 180°$). Suppose $c + a \geq 180°$. Since $AB + BD = AD = 180°$, then $c + a \geq AB + BD$ or (since $a = BC$ and $c = AB$) $c + a = AB + BC \geq AB + BD$ or $BC \geq BD$. Therefore in triangle BCD we have $\angle D \geq \angle BCD$ (as in Euclid I.18, here we have greater angles opposite greater sides). Since $\angle A = \angle D$ (by symmetry), and $\angle BCD = 180° - C$ (here C denotes $\angle BCA$ in triangle ABC), then $\angle A \geq 180° - C$, as claimed. Menelaus takes the converse as granted. If we suppose $c + a \geq 180°$, then the same argument produces $\angle A < 180° - C$, as claimed. ∎



Euclid's Proposition I.32 is replaced with Proposition I.11 of *Sphaerica*, which states:

Proposition I.11. For spherical triangle ABC, $A + B + C > 180°$.

Proof. Consider again triangle ABC in the figure above and let D be the pole opposite to A. Extend AB to AD and extend AC to AD, as in the figure. Introduce point E on CD as shown, so that $\angle DBE = \angle BDE$. Then $ED = EB$ since triangle BDE is isosceles. Hence $CE + EB = CE + ED = CD < AD = 180°$. So by Proposition I.10 (with $c = CE$ $a = EB$, in the notation of Proposition I.10, so that $c + a = CE + EB < 180°$), exterior angle $\angle ACB$ to triangle BCE at point C is greater than $\angle CBE$. That is, $C > \angle CBE$ (here C denotes $\angle BCA$ in triangle ABC). Since $D = \angle EBD$ (recall that triangle EBD is isosceles by construction, so these angles are equal) and A and D are equal angles (by symmetry), then $C + A > \angle CBE + \angle EBD = \angle CBD$. Adding angle B (here B denotes $\angle CBA$ in triangle ABC) to both sides of this last inequality gives $A + B + C > \angle CBD + B = \angle CBD + \angle CBA = 180°$ or $A + B + C > 180°$, as claimed. ∎

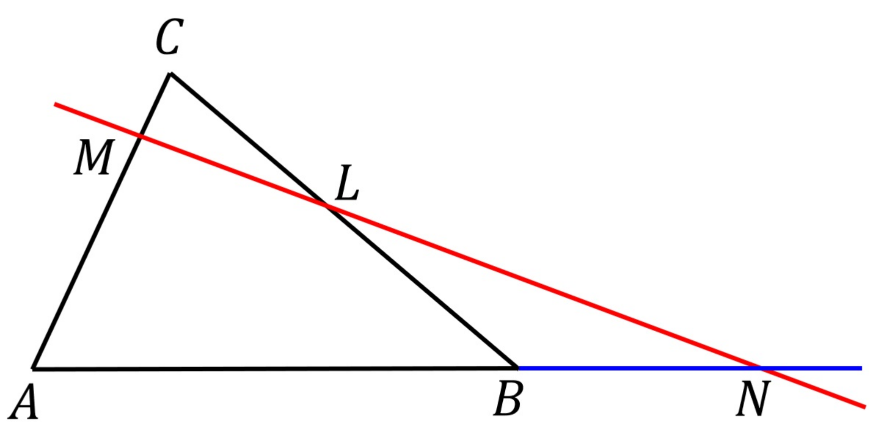
Proposition I.11 is not particularly surprising; we would not expect spherical triangles to behave exactly like plane triangles. However, Menelaus, his predecessors, and his successors would not think of this result as a clue to the existence of options to geometries other than Euclid's. It would be another 1700 or so years before the mathematical community would start to explore non-Euclidean geometry. Much of the delay is the result of there not being any known model of non-Euclidean geometry. "Hyperbolic geometry" would be the first type of non-Euclidean geometry to appear (a model for which is the Poincare disk which, in a sense, cannot be built as a surface in our three-dimensional space). In hyperbolic geometry, the angle sum of a triangle is *less than* 180°. The second type is "elliptic geometry." In elliptic geometry, the angle sum of a triangle is great than 180°, like in Proposition I.11. A model of one type of elliptic geometry is based on spherical geometry. . . the surface on which this geometry is done is not the full sphere, but half of the sphere to which some complicated connections have been imposed. This surface cannot by built in our three-dimensional space either! This will be explored more in Section 13.8. Non-Euclidean Geometry. See also my online supplements on Hyperbolic Geometry and A Quick Introduction to Non-Euclidean Geometry. The source for this note is Thomas Heath's *A History of Greek Mathematics, Volume 2* (Oxford: Clarendon Press, 1921), pages 260–265.

Note 6.5.F. Book II of Menelaus' *Sphaerica* established propositions of astronomical interest only. These were generalizations or extensions of propositions already presented in Book III of Theodosius' *Sphaerica*. Neither Book I nor Book II of Menelaus' work contains any trigonometry. His trigonometric results appear in

Book III. The Greeks did not use the terminology that we associate with trigonometry (i.e., sine, cosine, and tangent), but instead express things in terms of the chord function which we met in Note 6.5.C. Recall that we related the sine function and the chord function in that note as $\sin\alpha = \mathrm{crd}(2\alpha)/120$, because we considered (as Ptolemy does) a circle of radius 60. If we consider a circle of radius 1, then the since function and chord function are related as $\sin\alpha = \mathrm{crd}(2\alpha)/2$. With this convention, we then have $\cos\alpha = \sin(90^\circ - \alpha) = \mathrm{crd}(180^\circ - 2\alpha)/2$. Similarly,

$$\tan\alpha = \frac{\sin\alpha}{\cos\alpha} = \frac{\mathrm{crd}(2\alpha)}{\mathrm{crd}(180^\circ - 2\alpha)}.$$

Book III starts with what today is known as Menelaus' Theorem. There is both a plane version and a spherical version of the result. Menelaus assumed the plane version as well-known and used it to prove the spherical version. Eves states the plane version (on his page 176), but we need some preliminary explanation. Consider a triangle with a transversal (a line) which intersects each side (extending the sides as needed, and not allowing the intersections to coincide with vertices of the triangle). One possible configuration is the following:

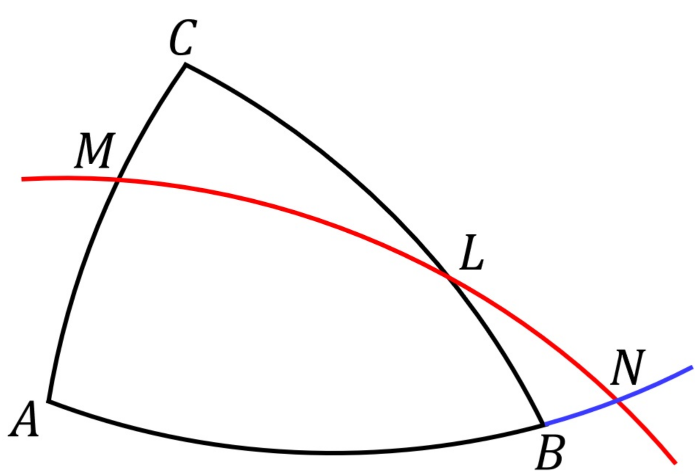


We consider quotients of the lengths of collinear line segments, but assign a sign to this quotient. In the configuration above, we have the quotient AN/NB is negative because N is not between A and B. The quotients BL/LC and CM/MA are positive because L is between B and C, and M is between A and C. With this convention, Eves states (see his page 176):

Menelaus' Theorem (Plane Version). If a transversal intersects the sides of BC, CA, AB of a triangle ABC in the points L, M, N, respectively, then

$$\left(\frac{AN}{NB}\right)\left(\frac{BL}{LC}\right)\left(\frac{CM}{MA}\right) = -1.$$

Notice that the negative sign in this theorem implies that a transversal can only intersect (internally) either two edges of the triangle (as in the figure above) or no edges of the triangle. As described in Note 6.5.E, we can measure segments of great circles on a sphere in terms of degrees (namely, the measure of the central angle in the sphere determined by the segment). Hence, we can apply the sine function or the chord function to such a segment. On possible configuration of a spherical triangle cut be a great circle transversal (similar to the plane case given in the above figure) is:



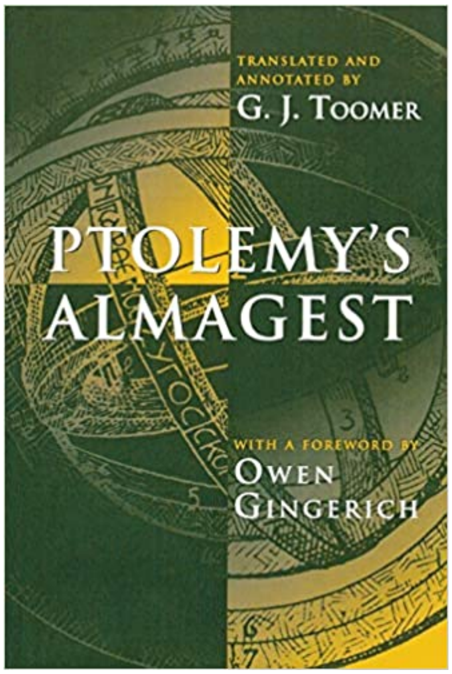
The version of Menelaus' Theorem in the spherical setting is stated by Eves as follows (see his page 177):

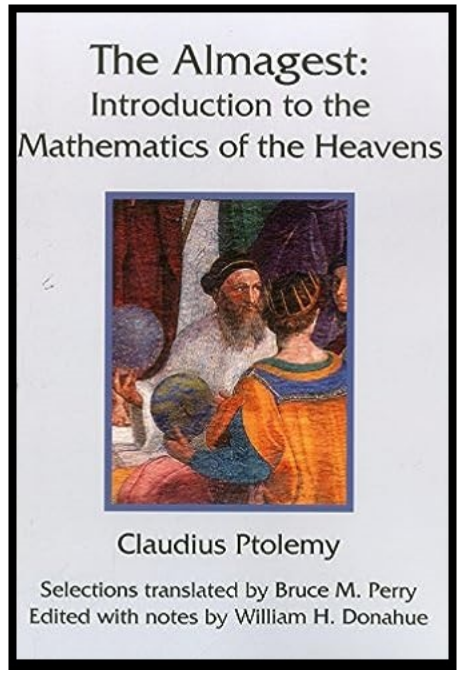
Menelaus' Theorem (Spherical Version). If a great circle transversal intersects the sides of BC, CA, AB of a spherical triangle ABC in the points L, M, N, respectively, then

$$\left(\frac{\sin AN}{\sin NB}\right)\left(\frac{\sin BL}{\sin LC}\right)\left(\frac{\sin CM}{\sin MA}\right) = -1.$$

Much of Book III involves applications of the spherical version of Menelaus' Theorem, "Proposition III.1." Many of the results are analogous to Euclid's results for plane triangles. For example, Proposition III.9 demonstrates that that for a spherical triangle, the great circles bisecting the three angles meet at a point (in fact, they must also meet at a second point, which is opposite the first). Propositions III.11 to III.15 relate to the same sort of astronomical problems addressed in Euclid's *Phaenomena*, Theodosius' *Sphaerica*, and Book II of Menelaus' *Sphaerica*.

Note 6.5.G. Claudius Ptolemy (circa 85 CE–165 CE) was a Greek astronomer and geographer whose geocentric theory as explained in his *Almagest* dominated astronomy for 1400 years (when it was replaced by heliocentrism, as given by Copernicus). Claudius Ptolemy is not to be confused with the members of the Ptolemaic Empire in Egypt (305 BCE to 30 BCE) mentioned in Section 5.1. Alexandria; see Note 5.1.C. Oddly, Ptolemy would not recognize the title *Almagest*. It was originally the *Mathematical Collection* (or the *Mathematical Syntaxis*). It came to be called "Great Collection." In Arabic, the article "*Al*" was added to the superlative "magestic" to give *Al-majisti*, which became *Almagest*. You may see it referred to by any of these, but in these notes we use the most common of these, the "*Almagest*." The Almagest is in print in English as *Ptolemy's Almagest*, Revised Edited Edition, translated and annotated by G. J. Toomer, Princeton University Press (1998), and also as *The Almagest: Introduction to the Mathematics of the Heavens*, Selections translated by Bruce M. Perry, Edited with Notes by William H. Donahue, Green Lion Press (2014).





Almagest consists of thirteen books. It contains observations and investigations of Hipparchus (190 BCE–120 BCE), and Ptolemy's Table of Chords is largely based on Hipparchus. "[I]t is questionable whether he himself contributed anything of grreat value except a definite theory of the motion of the five planets. . . " (Heath's *A History of Greek Mathematics, Volume II*, page 275). Book I includes preliminaries, explanations of the motions of the heavenly bodies, some spherical geometry, and his Tables of Chords. Book II continues Book I, giving consideration to observations at different latitudes such as the length of the day. Book III considers the length of the year and the motion of the sun as given by the epicycle hypothesis. Book IV covers the length of the month and the movement of the moon. Book V gives constructions of the astrolabe, and estimates the diameters of the sun and moon. Book VI continues the study of the motion of the sun and moon by considering solar and lunar eclipses and their periods. Books VII and VIII are about the fixed stars and the precession of the equinoxes. Books IX to XIII concentrate on the movements of the planets.

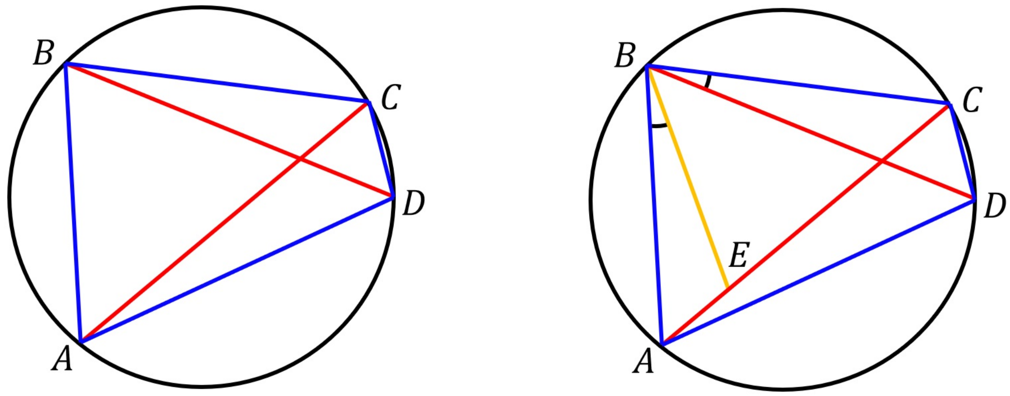
Note 6.5.H. The trigonometry Ptolemy presents in the *Almagest* is not new, but what is new is that he gives the minimum propositions necessary to establish what is given. Ptolemy divides the circle into 360 equal parts (or "degrees") and divides the diameter of the circle into 120 equal parts (requiring an adjustment to translate the chord function into the sine function). In Book I Section 10, "On the Size of Chords in a Circle," first the chord function is found for $36°$ and $72°$. This is equivalent to finding $\sin 18°$ and $\sin 36°$, as Note 6.5.F above. Second, in terms of the chord function, it is shown that $\sin^2\theta + \cos^2\theta = 1$. The next result is known as "Ptolemy's Theorem." Eves states this as (see page 177):

Ptolemy's Theorem. In a cyclic quadrilateral [that is, a quadrilateral inscribed in a circle], the product of the diagonals is equal to the sum of the products of the two pairs of opposite sides.

Ptolemy's Theorem is considered in Problem Study 6.9. It is to be shown there that the theorem implies the sum and difference formulas for the sine function and the half angle formula for sine. We now give a restatement and proof a Ptolemy's Theorem, as given in Heath's *A History of Greek Mathematics, Volume II* (see page 279).

Ptolemy's Theorem. Given a quadrilateral $ABCD$ inscribed in a circle, the diagonals being AC, BD, to prove that $(AC)(BD) = (AB)(DC) + (AD)(BC)$.

Proof. Introduce BE so that the angle ABE is equal to the angle DBC, and let BE meet AC at point E.



Adding angle EBD to angle ABE gives angle ABD, and adding angle EBD to angle DBC gives angle EBC. Since angles ABE and DBC are equal, then angles ABD and EBC are equal. By Euclid's Proposition III.21, angle BDA equals angle BCE (since both angles subtend arc AB). Therefore, triangle ABD is equiangular (i.e., "similar") to triangle EBC. Hence

$$BC : CE = BD : AD \text{ or } (BC)(AD) = (BD)(CE). \tag{1}$$

Since angles ABE and DBC are equal by construction, and angles BAE and CBD are equal, then triangle ABE is equiangular with triangle BCD. Hence

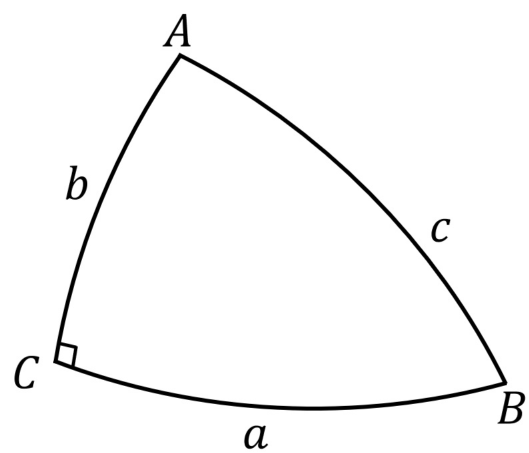
$$AB : AE = BD : DC \text{ or } (AB)(DC) = (BD)(AE). \tag{2}$$

Adding (1) and (2) gives $(AB)(DC) + (BC)(AD) = (BD)(AE) + (BD)(CE)$ or (since $(AE) + (CE) = (AC)$): $(AB)(DC) + (AD)(BC) = (AC)(BD)$, as claimed.

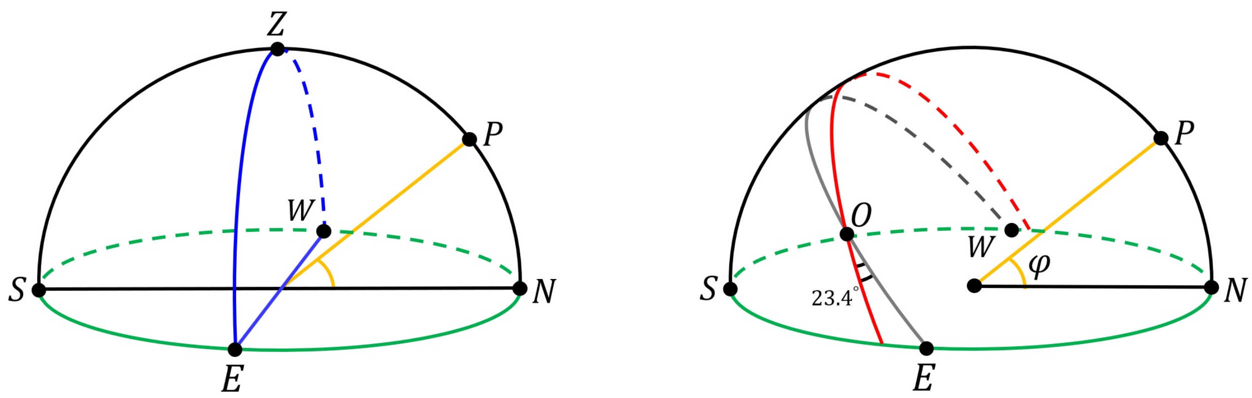
∎

Ptolemy estimates crd$(1/2^\circ)$ and then uses his addition formula to get estimates of the chord associated with each angle form $1/2^\circ$ to 180° in $1/2^\circ$ increments. This is how creates his "Table of Chords." Converting Ptolemy's chord function to sines generally yields values accurate to five decimal places. Ptolemy's results in on spherical trigonometry follow from Menelaus' Theorem (see Note 6.5.F above). For spherical triangle ABC with right angle C, the following relationships between the angles A, B, C and the lengths of the sides a, b, c (measured in terms of central angles, as described in Note 6.5.E above):

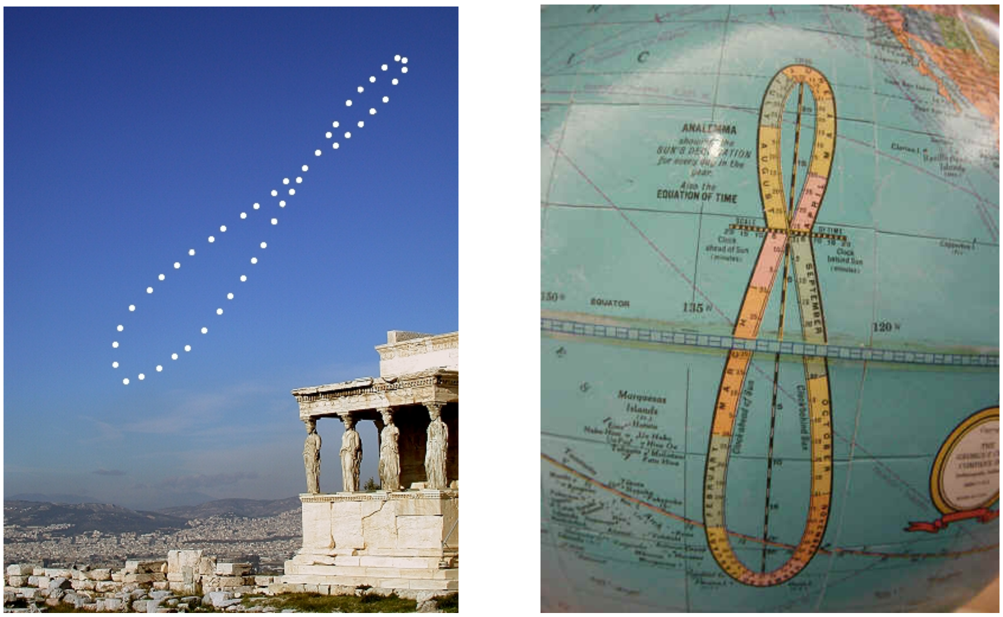
$$\sin a = \sin c \sin A, \quad \tan a = \sin b \tan A, \quad \cos c = \cos a \cos b, \quad \tan b = \tan c \cos A.$$



Note 6.5.I. Another work by Claudius Ptolemy is the *Analemma*. The goal of this work is to project points and arcs on the celestial sphere onto three mutually orthogonal planes (this is called *orthogonal projection*). The three planes are the plane containing the horizon, the plane containing the meridian (so that it contains the zenith point, the pole, the north most point, and the south most point on the celestial sphere), and the plane containing the zenith point, the east-point (i.e., the east most point on the celestial sphere), and the west-point.



In the figure above (left), the intersection of the horizon and the celestial sphere is in green, the meridian is given in black, and the the curve given in blue determines the "prime vertical" plane. The zenith is point Z and the pole is point P; the celestial sphere rotates through time about P and the North Star is appear very close to P in the sky). Angle φ represents the latitude of an observer (who is located at the intersection of the three orthogonal planes). Other great circles of interest are the celestial equator which intersects the green circle at points E and W and is in a plane perpendicular to the orange line (in grey in the figure above, right(, and the ecliptic (around which the sun, moon, and planets travel) which intersects the celestial equator at two points (called the *equinoctial* points; the sun is located at these points at the equinoxes, the celestial equator is given in red above and one of the equinocial points is labeled O with the other below the horizon) and is at an angle of 23.4° to the celestial equator (this angle results from the tilt of the Earth's axis). The *Analemma* is meant to be applied to the construction of sun dials, so that local time can be determined from the location of a shadow on the face of the sun dial. You may be familiar with the term "analemma" from astronomy or geography. In those settings it is a diagram showing the position of the sun in the sky as viewed from a specific position on the Earth at the same mean solar time (that is, the time given by a clock) every day. An analemma has the shape of a figure eight due to the fact that results from the sun's motion around the ecliptic (resulting in a north-south motion over the course of a year) and the east-west movement due to the eccentricity of Earth's orbit.

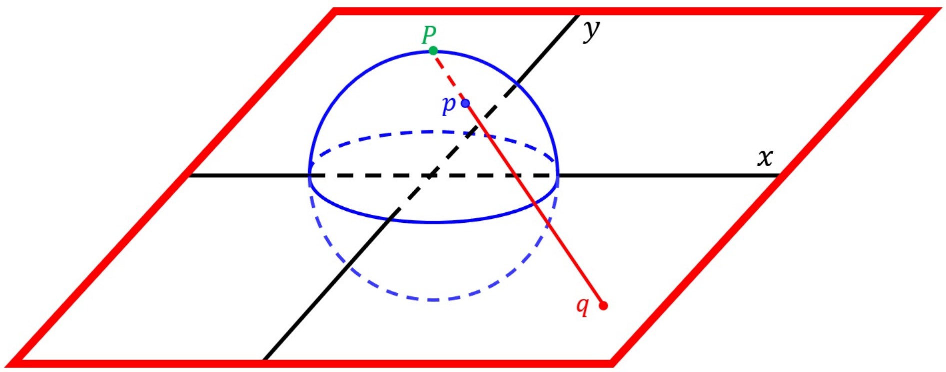


An image illustrating the analemma taken in Erechtheion, Athens, Greece during the year 2003, from the Stanford Solar Center (left). An analemma on an old globe (right) from the MathBabe.org webpage on the analemma (both pages accessed 9/11/2023).

The displacement of the sun from the centerline of the analemma indicates how far ahead or behind the time given by a sundial is from the time given by a clock. This allows the conversion of the time of day given by a sundial to the time as given by a clock.

Note 6.5.J. In *Planisphaerium*, Ptolemy gives a technique to create a flat map corresponding to the celestial sphere. The technique was probably known to the

Egyptians, and likely known to Hipparchus (190 BCE–120 BCE). However, the earliest known surviving work on the technique is Ptolemy's. The technique is called *stereographic projection* and involves projecting points on the celestial sphere onto the plane containing the celestial sphere. As illustrated below, the projection is based on point P on the celestial sphere (taken to be the north pole of the celestial sphere in the figure for ease of visualization).



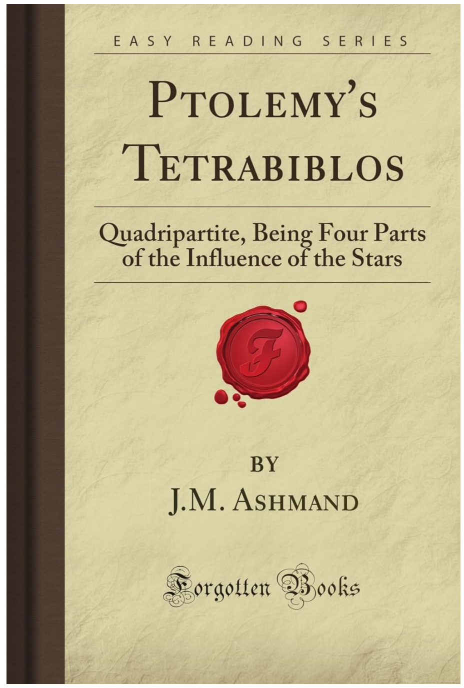
For given point p on the celestial sphere, a line through pole P and point p is used to find a point q in the plane containing the celestial equator. So point p is projected onto point q. Notice that this maps the lower half of the celestial sphere to the inside of the circle intersection of the sphere and the plane, maps the upper half of the celestial sphere to the outside of this circle, and fixes the circle itself (i.e., it fixes the equator of the celestial sphere). For the pole point P, there is no corresponding point in the plane under this projection (sometimes it is said that point P is "mapped to ∞"; see my online notes for Complex Analysis 1 [MATH 5510] on Section I.6. The Extended Plane and Its Spherical Representation for more on this). Also, this will lead to a lot of distortion for the stars near the north celestial pole P. In fact, Ptolemy chose to project through the south pole, so that the northern hemisphere of the celestial sphere would be less distorted on the flat surface. Ptolemy was aware that stereographic projection maps circles on the sphere to circles or lines in the plane, though he gave no formal proof of this. No versions in the original Greek survive, and *Planisphaerium* is only known through Latin translations of Arabic versions. Ptolemy also wrote *Optics* in five books. It too survives only through Latin translations of Arabic versions, though Book I and the end of Book V are missing. Book III is on the theory of mirrors, Book IV deals with concave and composite mirrors, and Book V covers refraction. Due to an underlying false assumption about reflection, there are some incorrect conclusions drawn.

Image from the MacTutor biography webpage of Claudius Ptolemy

Note 6.5.K. Ptolemy attempted to prove the Parallel Postulate (see Section 5.4. Content of the "Elements", Notes 5.4.B and 5.4.C) from the other results in Euclid's *Elements*. Of course, this was unsuccessful! We know about this work because

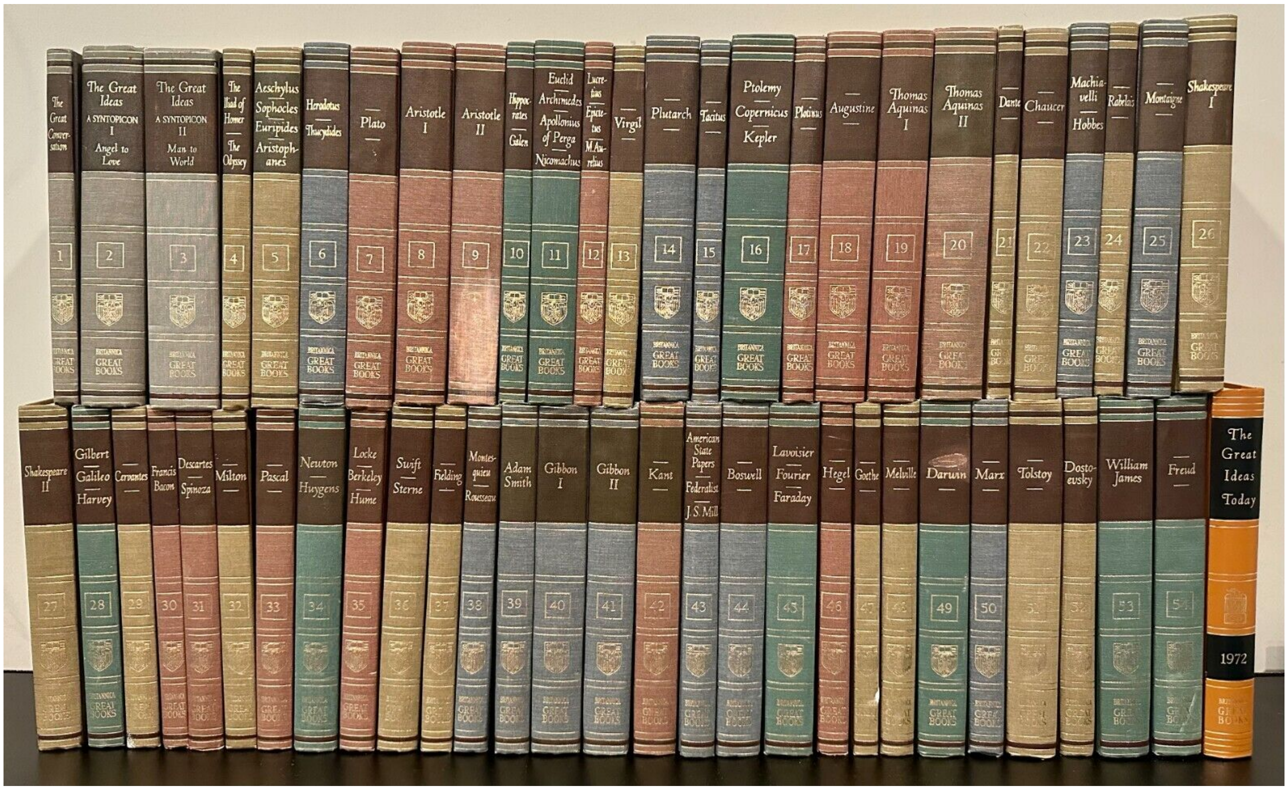
Proclus (circa 411 CE–April 17, 485 CE) in his commentary on Book I of Euclid's *Elements*. According to Proclus, Ptolemy first gives a proof of Euclid's Proposition I.28 ("If a straight line falling on two straight lines makes the exterior angle equal to the interior and opposite angle on the same side, or the sum of the interior angles on the same side equal to two right angles, then the straight lines are parallel to one another.") and then a failed proof of Euclid's Proposition I.29 ("A straight line falling on parallel straight lines makes the alternate angles equal to one another, the exterior angle equal to the interior and opposite angle, and the sum of the interior angles on the same side equal to two right angles."). However, the proof of Proposition I.29 requires the Parallel Postulate (which is why Ptolemy's proof is wrong). In fact, Proposition I.29 is the first proposition of the *Elements* to require the Parallel Postulate. Ptolemy had assumed that, through a point not on a line, there exists exactly one line parallel to the given line. Today, this is known as Playfair's Axiom after John Playfair (March 10, 1748–July 20, 1819), and it is equivalent to the Parallel Postulate (so this is where Ptolemy's proof has its flaw). We'll see more about Playfair's Axiom (or, under the assumption of the Parallel Postulate, Palyfair's *Theorem*) in Section 13.8. Non-Euclidean Geometry; see also Supplement. A Quick Introduction to Non-Euclidean Geometry.

Note 6.5.L. As a final observation about Ptolemy, we consider his contributions to astrology. In Ptolemy's time, astrology and astronomy were closely related. Ptolemy's *Almagest* considered the prediction of the location of the planets in the sky (recall that the sun and moon counted as planets, since they move with respect to the "fixed stars"). In a separate work, *Tetrabiblos* or "Quadrapartite being Four Books of the Influence of the Stars" meant as a companion book to the *Almagest*, he presented his views on the influence of the planets on human affairs. We should comment that Aristotle (384 BCE–322 BCE) ascribed changes on Earth to the effects of the heavens. For example, in his *Meteorology* he starts with the assumption that all power of change in nature comes from the heavens. Of course *some* changes on Earth are the result of the positions of the "planets"; anyone with a familiarity of tables of tides knows that high and low tides are affected by location of the sun and moon. In addition, the seasons are definitely related to the location of the sun relative to fixed stars (a fact realized and celebrated in prehistory). The idea of heavenly effects on Earthly matters in terms of the locations of the stars and planets and their predicted influence is the topic of interest in astrology (sometimes the five naked-eye planets in the night sky, other than the sun and moon, are called "wandering stars"). It remained a part of "science" throughout the middle ages up to the time of Kepler (though today it is taken to be pseudo-science). Kepler and his views on astrology are considered in Section 9.7. Kepler. So knowing where the planets would appear in the night sky became something of practical importance in the minds of many. Most people today know their astrological "sign." Your sign is an artifact of the time of Ptolemy. Due to the "wobble" of the Earth's axis over time, the location of the sun in a constellation in Ptolemy's time is different from what it is today; the constellations of the zodiac appear to drift through time. The wobble is formally called *axial precession*; it results in about a 1° change over the span of a human lifetime and so is not really noticeable to the casual observer. As a result, the old positions of 2000 years ago do not coincide with modern-day positions and are off by roughly one constellation (which corresponds to about one month). For example, if you were born on Christmas day, December 25 (a few days after the winter solstice in the northern hemisphere), then the sun is in the constellation Sagittarius but your "sign" is Capricorn. The solution reached by astrologers was to simply distinguish between your "sign" and the "constellation" in which the sun appears at the time of your birth. This is all further complicated by the fact that the ecliptic passes through the constellation Ophiuchus and the sun is in this constellation between November 30 and December 17 (give or take a day)... Of other significance is the location of the planets in the sky at a particular time and place. The half of the ecliptic (or zodiac) which is above the horizon is partitioned into six "houses." Three of these lie between the eastern horizon and the meridian (these are the *ascendant* houses) and the other three lie between the meridian and the western horizon (these are the *descendant* houses). The houses remain in the same locations, but as the Earth rotates the planets and constellations in the houses change. All of this movement and geometry, required careful calculation. This is where early mathematical astronomy met the astrology of the time. In *Tetrabiblos* Ptolemy acknowledges the uncertainties present in this work and that scholars should "never compare its perceptions with the sureness of the first, unvarying science [astronomy as presented in *Almagest*]" (Falk's *The Light Ages*, page 184). Given the ongoing interest in astrology (an inexplicable interest, to your instructor), it is not surprising that *Tetrabiblos* is still in print. One version (of several) is Tyler Ashmand's *Ptolemy's Tetrabiblos: Quadripartite, Being Four Parts of the Influence of the Stars* (reprinted by Forgotten Books, 2008):



It can also be read online on the ForgottenBooks.com website; accessed 10/6/2023). The source for this note is Seb Falk's *The Light Ages: The Surprising Story of Medieval Science* (W. W. Norton, 2020) pages 183, 184, and 186.

Note 6.5.M. A number of sources for English translations of historically influential mathematics (and astronomy) works have been given so far. Many of them (in particular, those currently in print) are available through Dover Publications (such as Heath's translation of Euclid's *Elements* mentioned in Section 5.3. Euclid's "Elements" in Note 5.3.K). Another widely used source in the 20th century (before the internet and the presence of online versions) was the *Great Books of the Western World* series published by Encyclopædia Britannica, Inc. in 1952.



This image is from an E-Bay auction (accessed 9/8/2023).

This was a 54 volume set of books that eventually sold a million sets. Anecdotally, it seems that many public libraries had copies of the set. Of particular interest to this section of notes is the fact that Volume 16 included translations of Ptolemy's *Almagest*, Copernicus' *on the Revolutions of Heavenly Spheres*, and Kepler's *Epitome of Copernican Astronomy* (Books IV and V) and his *The Harmonies of the World* (Book V). In addition, the following volumes are relevant to the history of mathematics:

Volume 11. This presents Euclid's *Elements* (it gives the Heath translation, but with no commentary), several works of Archimedes (including *On the Sphere and Cylinder*, *The Quadrature of the Parabola*, and *The Method*), Apollonius' *On Conic Sections*, and Nichomachus; *Introduction to Arithmetic*).

Volume 28. This includes Galileo's *Dialogues Concerning the Two New Sciences*.

Volume 31. This has several works of Descartes, including *The Geometry*.

Volume 34. This gives two of Newton's works, *The Mathematical Principles of Natural Philosophy* and *Optics*.

Volume 45. This includes Lavoisier's *Elements of Chemistry*, Fourier's *Analytical Theory of Heat*, and Faraday's *Experimental Researches in Electricity*.

Also included are works by Homer (Volume 4), Plato (volume 7), Aristotle (Volumes 8 and 9), Hippocrates (Volume 10), Chaucer (Volume 22), Shakespeare (Volumes 26 and 27), Milton (Volume 32), Kant (Volume 42), Darwin (Volume 49), and Freud (Volume 54). Criticism of the series has called it "a celebration of European men, ignoring contributions of women and non-European authors." This quote (with references) and the other information of this note is from the Wikipedia webpage on the Great Books of the Western World (accessed 9/8/2023). A second edition was released in 1990, with the addition of Volumes 54 through 60. Some of the criticism was addressed by the addition of work by Jane Austin (*Emma*), Willa Cather (*A Lost Lady*), and Virginia Woolf (*To the Lighthouse*). The second edition also included work by philosopher/mathematician Alfred North Whitehead (*An Introduction to Mathematics*) and G. H. Hardy (*A Mathematician's Apology*) in Volume 56 (along with work by Poincaré, Plank, Einstein, Heisenberg, and Schrödinger) .

Revised: 10/24/2023