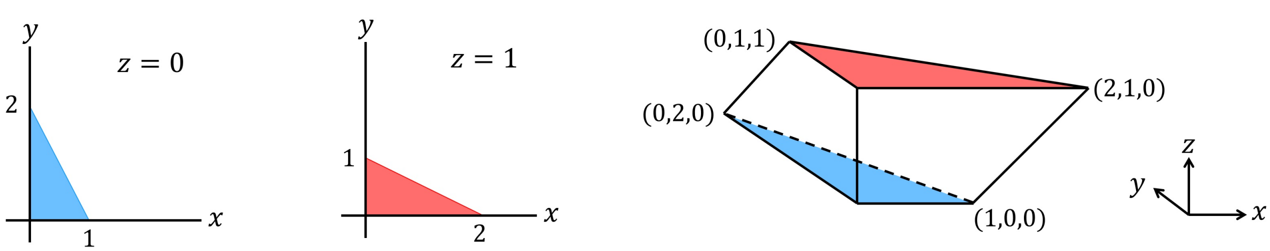
Supplement. The Volume of a Prismatoid

Note. In this supplement to 6.6. Heron. We consider Heron's rectangular prismatoid formula and show that it implies the general prismatoid formula. This is based on Eves' Problem Study 6.11(g), which has us show that these two formula are equivalent. Show that the general formula implies Heron's formula is left as an exercise. However, the converse seems rather involved and it is considered here (with one remaining concern about convexity; see Note 6.6S.G).

Note 6.6S.A. In Problem Study 6.11(g) we have the following.

> A *prismatoid* is a polyhedron all of whose vertices lie in two parallel planes. The two faces in these parallel planes are called the bases of the prismatoid, the perpendicular distance between the two planes is called the *altitude* of the prismatoid, and the section parallel to the bases and midway between them is called the *midsection* of the prismatoid. Let us denote the volume of the prismatoid by V, the areas of the upper base, lower base, and midsection by U, L, M, and the altitude by h, as indicated in Figure 55. In books on solid geometry, it is shown that
>
> $$V = \frac{h(U + L + 4M)}{6}.$$
>
> In Book II of the *Metrica*, Heron gives, as the volume of a prismatoid having similarly oriented rectangular bases with corresponding pairs of dimensions a, b, and c, d,
>
> $$V = h\left[\frac{(a + c)(b + d)}{4} + \frac{(a - c)(b - d)}{12}\right].$$

Eves' has not clearly defined a prismatoid here. There is also the requirement that the faces are either triangles (when the two sides between the bases share a point of a base) or trapezoids (when the two sides between the bases do not share a point on either base). A complete definition is given in W. S. Kern and J. R. Bland's *Solid Mensuration with Proofs*, 2nd Edition (NY: John Wiley & Sons, 1938), Section 30, page 75. Simply considering polyhedra in parallel planes does not insure that the faces are trazezoids. Consider the two triangles in the following figure (left and center). They determine a polyhedron as given in the figure (right). Two of the faces are trapezoids (those in the xz-plane and the yz-plane). However, the back face contains the four points $(1, 0, 0)$, $(0, 2, 0)$, $(0, 1, 1)$, and $(2, 1, 0)$, and these points are not coplanar. So the back face is not a trapezoid (it is not even flat), and the upper edge is not parallel to the lower edge (since these edges are not coplanar).



Problem Study 6.11(g) involves showing that these two equations are equivalent. We leave it as an exercise to show that the general prismatoid formula implies Heron's formula. Here, we show that Heron's formula implies the general prismatoid formula.

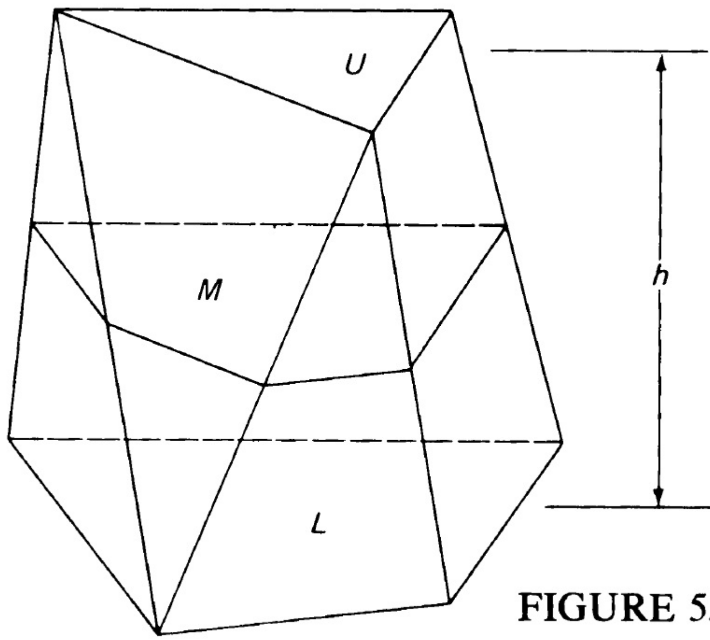


FIGURE 55

Note 6.6S.B Heron's formula applies to special prismatoids, as given in Figure 6.6.A. We can take $c = d = 0$ (or, if you prefer, take limits $\lim_{c\to 0^+}$ and $\lim_{d\to 0^+}$) to get that the volume of a pyramid with a rectangular base and altitude h is

$$V = h\left(\frac{ab}{4} + \frac{ab}{12}\right) = h\left(\frac{3ab}{12} + \frac{ab}{12}\right) = hab/3.$$

Notice that if we cut such a pyramid in half (through the apex and two points on opposite corners of the rectangular base), then we find that a pyramid with a right-triangle-base, where the legs of the triangle are lengths a and b and the altitude is h, has a volume of $hab/6$. Notice this is $h/3$ times the area, $ab/2$, of the right triangle.

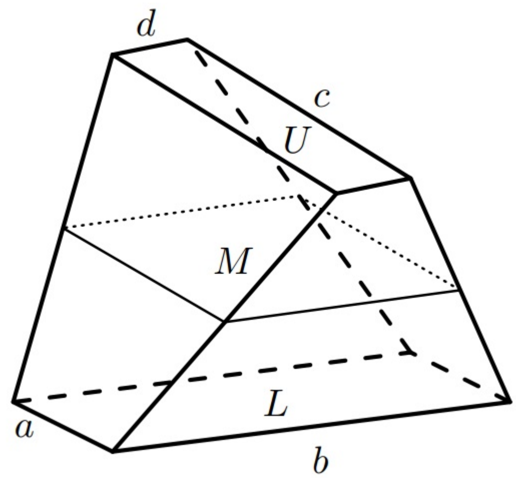


Figure 6.6.A. Based on an image from Chuck Garner's The MATHEMATICAL WORKS OF Dr. G. (this is an excellent source of several history of math problems; accessed 5/8/2024)

Now every triangle (obtuse of acute) can be partitioned into two right triangles, so that any triangle has an area equal to the sum or difference of the area of two right triangles. See Figure 6.6.B.

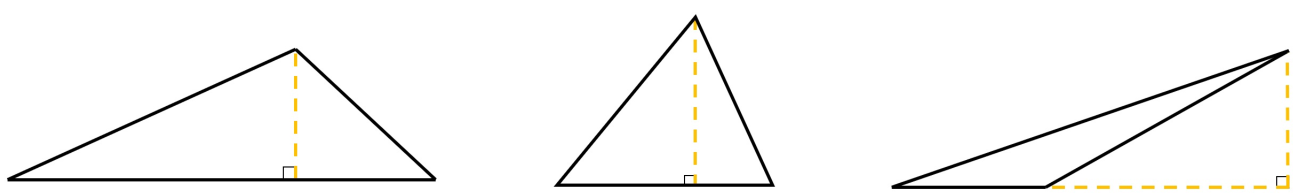


Figure 6.6.B. Expressing a triangle as the sum or difference of right triangles

So a pyramid with *any triangular base* and altitude h has a volume equal to the sum of the volumes of two right-triangle-base pyramids with altitude h. Hence, any pyramid with a triangular base and altitude h has a volume equal to the $h/3$ times the area of the base. Next, any polygon can be partitioned into triangles, so the volume of *any pyramid with a polygonal base* and altitude h has a volume equal to the sum or difference of the volumes of pyramids with the triangular bases and altitude h. That is, the volume an any pyramid with a polygonal base if the $h/3$ times area of the base. We'll need one more result based on Heron's formula.

Note 6.6S.C. But first, consider a prismatoid with lower base of area B, upper base of area U, and midsection of area M. For the sake of illustration, let the lower base be a polygon with vertices H, I, J, K, N, let the upper base have vertices E, F, G, and let R and S be two vertices at the ends of the same edge of the polygon of the midsection. See Figure 6.6.C.

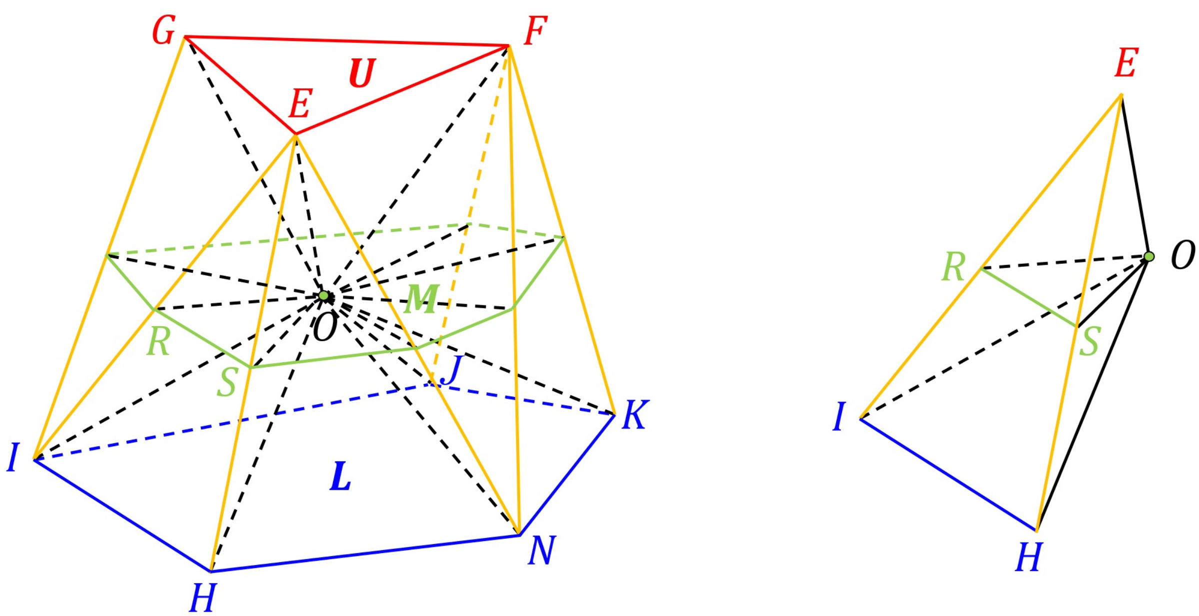


Figure 6.6.C. Based on an image from W. S. Kern and J. R. Bland's *Solid Mensuration with Proofs*, 2nd Edition (NY: John Wiley & Sons, 1938); see page 76

Let O be any point in the interior of the midsection. Introduce line segments from O to every vertex of the polygons for the lower base, upper base, and midsection (represented in black in the figure). Notice that this implies that the midsection must be a convex set! This allows us to partition the prismatoid into solids, some of which are pyramids. The base of one of the pyramids is the upper base of the prismatoid. This pyramid has altitude $h/2$ and so has volume $U(h/2)/3 = hU/6$. The base of another pyramid is the lower base of the prismatoid and similarly has volume $hL/6$. The remaining pyramids have bases which are triangular faces of the prismatoid. Consider the pyramid in the figure with base EHI (Figure 6.6.C, right). Triangles ERS and EIH are similar and the ratios of the lengths of their edges is $1:2$. So the ratio of the areas of these triangles is $1:2^2 = 1:4$ by Proposition VI.19 of Euclid's *Elements*. The altitude of any pyramid is the distance from the apex to the plane containing the base. Pyramids $OERS$ and $OEIH$ have their bases in the same plane, and so have the same altitude. So the volume of these pyramids are also in a ratio of $1:4$; that is, the volume of pyramid $OERS$ is $\frac{1}{4}$ of the volume of pyramid $OEIH$. We can also treat triangle ORS as the base of pyramid $OERS$ (this is the case because the face EIH is a triangle) and then the altitude is $h/2$. So the volume of pyramid $OERS$ is also $\frac{1}{3}$ of the area of triangle ORS times $h/2$. We now have that the volume of $OERS$ is $h/6$ times the area of triangle ORS, from which

$$\text{the volume of } OEIH \text{ is } 4h/6 = 2h/3 \text{ times the area of triangle } ORS. \qquad (*)$$

Note 6.6S.D. If a face of the prismatoid is a trapezoid, then we must take a slightly different approach. See Figure 6.6.F of pyramid $OFGIJ$. Again starting with Heron's formula for the volume of a prismatoid with similarly oriented rectangular bases:

$$V = h\left[\frac{(a+c)(b+d)}{4} + \frac{(a-c)(b-d)}{12}\right],$$

we take $d = 0$ (or, if you like, $\lim_{d\to 0^+}$) to get the area of solid with a rectangular lower base and an upper base of a line segment of length c in the same orientation as the sides of length a (we can take the line segment as a "degenerate" polygon and the solid as a degenerate prismatoid; see Figure 6.6.D):

$$V = h\left[\frac{(a+c)b}{4} + \frac{(a-c)b}{12}\right] = h\left(\frac{3ab+3bc+ab-ac}{12}\right)$$
$$= h\left(\frac{4ab+2bc}{12}\right) = \frac{h(2ab+bc)}{6} = \frac{h(2(ab)+bc)}{6}.$$

Notice the form of this expression; it involves the quantity (ab) which represents the area of the rectangular base.

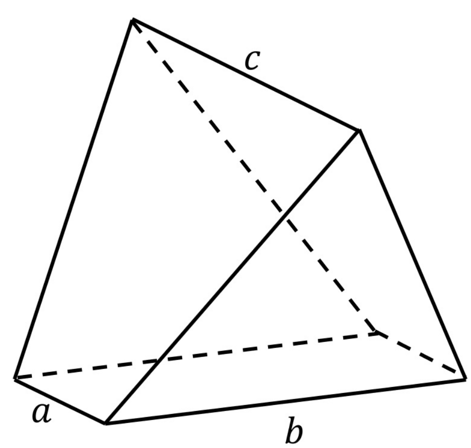


Figure 6.6.D. A degenerate prismatoid with upper base a line segment

We can partition this solid into two solids by cutting the base with a plane through opposite corners of the rectangular base and through an endpoint of the upper line segment "base," is shown in Figure 6.6.E.

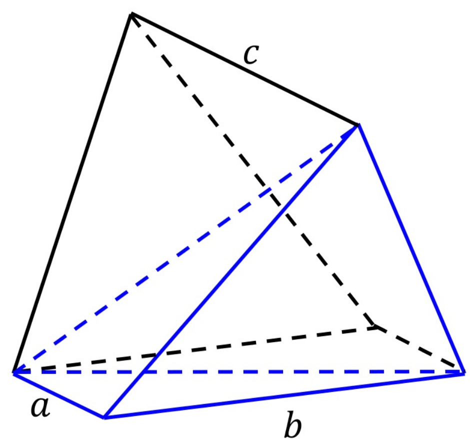


Figure 6.6.E. Partitioning the degenerate prismatoid of Figure 6.6.D into a pyramid (in blue), and a degenerate prismatoid with a right-triangle base

The volume of such a solid with lower base a right triangle with legs a and b, and upper "base" a line segment of length c is the volume of the original degenerate prismatoid with rectangular base minus the pyramid with right triangle base and so is

$$V = \frac{h(2ab+bc)}{6} - \frac{h}{3}\left(\frac{ab}{2}\right) = \frac{h(ab+bc)}{6} = \frac{h(2(ab/2)+bc)}{6}.$$

Notice the form of this expression; it involves the quantity $(ab/2)$ which represents the area of the right triangle base. Similar to the computation for volumes of pyramids above, we can express any triangle as a sum or difference of right triangles as illustrated in Figure 6.6.B, so that a degenerate prismatoid with a triangular base can be expressed as the sum or difference of a degenerate prismatoid with a right triangle base and a pyramid with a right triangle base. Adding or subtracting the volumes yields a formula of the form $V = \dfrac{h(2(A)+bc)}{6}$ where A is the area of the triangular base of the degenerate prismatoid.

Note 6.6S.E. Consider again the solid of Figure 6.6.C (left). The trapezoidal face $FGIJ$ produces the pyramid of Figure 6.6.F, where we have introduce points V, Z,and W in the midsection (notice that Z lies on the line VW but may not necessarily lie on face $FGIJ$; this does not affect our computations since we simply use (OZ) as the altitude of the pyramid). In terms of the points labeled we have, by treating the volume the pyramid as the sum of two degenerate prismatoids both with triangular base (an upper one with degenerate "base" (FG) and a lower one with degenerate "base" (IJ)) and altitude $h/2$, that the volume is

$$V = \frac{(h/2)(2(A) + (OZ)(FG)}{6} + \frac{(h/2)(2(A) + (OZ)(IJ)}{6}$$
$$= \frac{h}{12}(4A + (OZ)((FG) + (IJ))$$

where A is the area of triangle OVW. Now VW lying on the midsection satisfies $(VW) = ((FG) + (IJ))/2$ or $2(VW) = (FG) + (IJ)$. The volume of the pyramid is then

$$V = \frac{h}{12}(4A + (OZ)((FG) + (IJ)) = \frac{h}{12}(4A + (OZ)(2(VW))$$
$$= \frac{h(2A + (OZ)(VW)}{6} = \frac{h(2A + 2A}{6} = \frac{2h}{3}A,$$

since $(OZ)(VW)$ is the product of the base and height of triangular base OVW. That is, the volume of $OFGIJ$ is $2h/3$ times the area of triangle OVW. $(**)$

Note 6.6S.F. We now have from $(*)$ and $(**)$ that each pyramid in the figure with apex O has a volume equal to $2h/3$ times the area of the triangular cross section of the pyramid in the midsection. Summing over all such pyramids gives a total volume of $2hM/3$ (recall that the midsection is of area M). Adding in the pyramids with bases L and U, we get the total volume of the prismatoid is:

$$V = \frac{hU}{6} + \frac{2hM}{3} + \frac{hL}{6} = \frac{h}{6}(U + 4M + L).$$

That is, Heron's formula on prismatoids with rectangular bases implies the general formula for the volume of a prismatoid, as claimed.

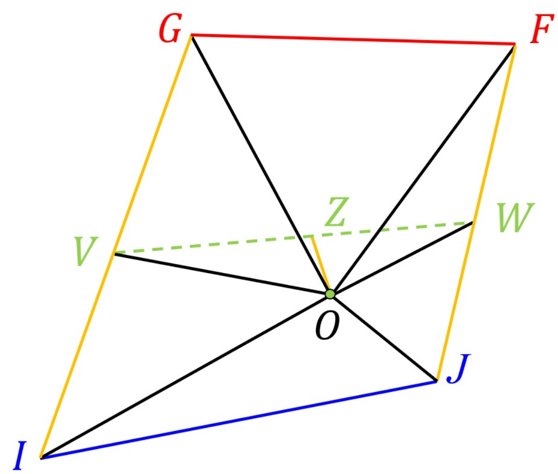


Figure 6.6.F.

This argument is based on Kern and Bland's *Solid Mensuration with Proofs*, pages 76 and 77. Their presentation does not address the "degenerate prismatoids." Instead they address the pyramids that result from triangular faces of the solid (as we did with pyramid $OEIH$) and simply state (on their page 77): "In a similar manner the volume of each of the remaining portions of the figure... can be shown to be equal to $4 \cdot \frac{h}{6}$ times the area of the mid-section included in it."

Note 6.6S.G. The only lingering concern (of your humble instructor) is the choice of point O as *any* interior point of the midsection. Only when the midsection is bounded by a convex polygon is it possible to join point O to all of the vertices of this polygon with line segments that lie within the midsection (thus partitioning the midsection into triangles, as needed). There are certainly prismatoids with non-convex midsections; simply take a traditional non-convex 5-sided star (or any convex set for that matter) as both the lower and upper bases of the prismatoid and join corresponding vertices. This will give a midsection that is also a non-convex 5-sided star. Perhaps the prismatoid can be decomposed into convex components. . .

Note. According to Kern and Bland (see their page 76), the generalized formula was first stated by Newton in 1711 and first published by James Sterling in his *Methodus Differentialis* in 1730. A proof of the generalized formula is given by Sarah Uhlig in "The Generalized Prismatoidal Volume Formula," *Pi Mu Epsilon Journal*, **8**(7), 455–458 (FALL 1987). It is based on integration, the assumption that the cross sectional area function is a polynomial of degree three or less, and Simpson's Rule. For information on Simpson's Rule, see my online notes for Calculus 2 (MATH 1920) on Section 8.6. Numerical Integration and notice Theorem 1b. Another reference, that also includes some history, is A. Day Bradley's "Prismatoid, Prismoid, Generalized Prismoid," *The American Mathematical Monthly*, **86**(6), 486–490 (June/July, 1979). Both of these papers can be found through JSTOR (though it may require your ETSU username and password; accessed 5/9/2024).

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