6.6. Heron

Note. Heron of Alexandria, also known as "Hero," is important for his contributions to geometry and mechanics. There is a surprising amount of uncertainty over the time in which he lived. Estimates range from about 150 BCE to 250 CE. The MacTutor biography page for Heron (accessed 5/7/2024) gives his dates as 10 CE–75 CE, but this is largely speculative. Thomas Heath in his *A History of Greek Mathematics, Volume II: From Aristarchus to Diophantus* (Clarendon Press, Oxford, 1921) devotes nine pages (pages 298 to 306) to the controversies over Heron's dates; this work is our main reference for these section of notes and we refer to it simply as "Heath's *History, Volume 2*."

From the MacTutor biography page for heron of Alexandria (accessed 5/7/2024); this is credited to a 1688 German translation of Heron's *Pneumatics*

Note 6.6.A. Heron concentrated more on practical applications of mathematical ideas the he did on mathematical completeness. His list of work includes the following (given in no particular order), which are known in Greek, and were translated or edited as indicated:

1. The *Metrica*, which was discovered in 1896 in an 11th or 12th century manuscript found in Constantinople.

2. *On the Dioptra* came out in an Italian version in 1814. It includes a description of a wheeled device to measure distance.

3. The *Pneumatica* (in two books) appeared in a Latin translation around 1575. This is available in English on the Library of Congress webpage (accessed 5/9/2024). It is *The Pneumatics of Hero of Alexandria, From the Original Greek*, translated by Bennet Woodcroft (London: Taylor Walton and Makerly, 1851).

4. *The automaton-theatre*, appeared in a 1589 Italian version. This is available on the Internet Archive webpage (accessed 5/9/2024) in Greek and German as *Herons von Alexandria, Druckwerk und Automatentheater*, by Wilhelm Schmidt (Leipzig: Druck und Verlag von B. G. Teubner, 1899).

5. *Belopoeïca* (on the construction of engines of war) was edited in 1616.

6. Geometrical works titles are *Definitions*, *Geometria*, *Geodaesia*, *Stereometrica*, *Mensurae*, *Liber Geeponicus*. Heiberg has translated these into German.

7. *Mechanics* in three books survives in fragments. This is available in French on the Internet Archives webpage as Baron Carra de Vaux's *Les Mècaniques ou L'Élévateur de Héron D'Alexandrie* (Paris: Imprimerie Nationale, 1894).

8. *Commentary on Euclid's Elements* in known from fragments of Greek work, but also from an Arabic commentary by an-Nairīzī (circa 865–circa 922).

More work by Heron is available in German in *Heronis Alexandrini Opera quae supersunt omnia* [All the surviving works of Hero of Alexandria] , a five volume collection. It includes work in Latin, with German translations throughout. Editors are Johan Heiberg, Herman Schöne, Ludwig Nix, and Wilhelm Schmidt. This was published in Leipzig by B. G. Teubner between 1899 and 1914. Links to images of these volumes are available on the HathiTrust.org website (accessed 5/9/2024). Volume 1 contains *Pneumatica* and *The automaton-theater* translated by Schmidt. Volume 2 contains the fragments of *Mechanics* translated by Nix and Schmidt. Volume 3 contains *Metrica* and *On the Dioptra* translated by Schöne. Volumes 4 and 5 contain the geometrical works translated by Heiberg. It is surprising that more of this work has not been translated into English. These volumes are likely the main source for Heath's *History, Volume 2* material on Heron. We now turn our attention to the works of Heron of a mathematical content.

Note 6.6.B. Heron's *Commentary on Euclid's Elements* is known to go at least as far as Proposition VIII.27. Heath states that (page 310): "Speaking generally, Heron's comments do not appear to have contained much that can be called important." Heron adds cases to several proofs due to the way a referenced figure might be drawn (such as locating chords of a circle on the same or opposite sides of a given diameter). He also gives a number of alternative proofs, particularly in Book III. For example, Euclid's Proposition III.13 states "A circle does not touch another circle at more than one point whether it touches it internally or externally." Heron gives an alternative proof of this based on a lemma he proved concerning the fact that a straight line cannot meet a circle in more than two points. In his commentary on Book I, Heron avoids extending a straight line, where Euclid does not. This addresses potential objections that might arise as to whether the needed space is available to make the extension of the line. This type of concern is dealt with in a modern axiomatic approach to geometry by giving a "Ruler Postulate" which implies a one-to-one correspondence between the real numbers and the points on any given line, so that there is "always room for more!" See my online notes for Introduction to Modern Geometry (MATH 4157/5157) on Section 2.4. The Measurement of Distance and notice Postulate 11. Heron gives converses of some of Euclid's propositions and extensions of others. An important extension is of Euclid's Proposition III.20: "In a circle the angle at the center is double the angle at the circumference when the angles have the same circumference as base." In other words, in a circle, a central angle has a measure twice that of an inscribed angle when the angles are subtended by the same arc. Euclid gives a proof for the inscribed angle at most a right angle, and Heron gives a proof allowing the inscribed angle to be bigger than a right angle. This results in an easier proof of Proposition III.22: "The sum of the opposite angles of quadrilaterals in circles [that is, quadrilaterals inscribed in circles] equals two right angles." In fact, this proposition is used in Problem Study 6.11(d) part (3) which gives Heron's formula for the area of a triangle in terms of the lengths of the three sides.

Note 6.6.C. In the preface to *Definitions*, Heron explains that he bases his approach on Euclid and that he is giving material for a good understanding of Euclid and other works of geometry. Heath describes the historical significance of this work as (page 316): "The *Definitions* are very valuable from the point of view of the historian of mathematics, for they give the difference alternative definitions of the fundamental conceptions. . . ” He gives 132 definitions in total. He starts with definitions of point, straight line, circular line, spiral shape (that is, the Archimedean spiral), surfaces, solid body, and angles (plane and solid). Rectilinear figure, various kinds of triangles and quadrilaterals, polygons, perpendiculars, and parallels are next. Then sphere, cone, acute-angled cone, obtuse cone, right-angled cone, ellipse, parabola, and hyperbola are defined. He defines several special solids, including rectilinear sold figures, pyramids, the five regular solids, prisms, parallelepipeds, and the semi-regular Archimedean solids. Equality and similarity of these objects are defined. Heron concludes with ratios of magnitudes, and commensurable and incommensurable magnitudes.

Note 6.6.D. Heron's work *Metrica* involves measurement (or "mensuration") and is the most historically important, since it is the most complete in its original form. It is also more theoretical than much of Heron's other work. It is mentioned in Eutocius' (circa 480–circa 540) commentary on Archimedes' *Measurement of a Circle* and this was the only evidence for it, until it was discovered by R. Schöne in Constantinople. Eutocius mentions that Heron has a technique for approximating the square root of a non-square number. We'll see more of this below, including why one would connect this to the work of Archimedes. *Metrica* consists of three books. In Book I, areas are considered for triangles, trapezoids, rhombi, quadrilaterals with one right angle, regular polygons up to those of 12 sides, segments of circles, ellipses, parabolic segments, and surfaces of cylinders, cones, and segments of spheres. In Book II, volumes are considered for parallelepiped, prism, pyramid and a frustum,

frustum of a cone, segment of a sphere, the solid "hoof" of Archimedes' *Method* (see Note AM2.F of Supplement. Archimedes' Method, Part 2), and the intersection of two cylinders as considered in Archimedes' *Method* (see Note AW.B of Supplement. The Content of Archimedes' Work, Part 1). Book III considers the division of figures into two parts that are in a given ratio; the figures considered are plane figures, pyramids, a cone and frustum, and a sphere. We next consider this content in more detail.

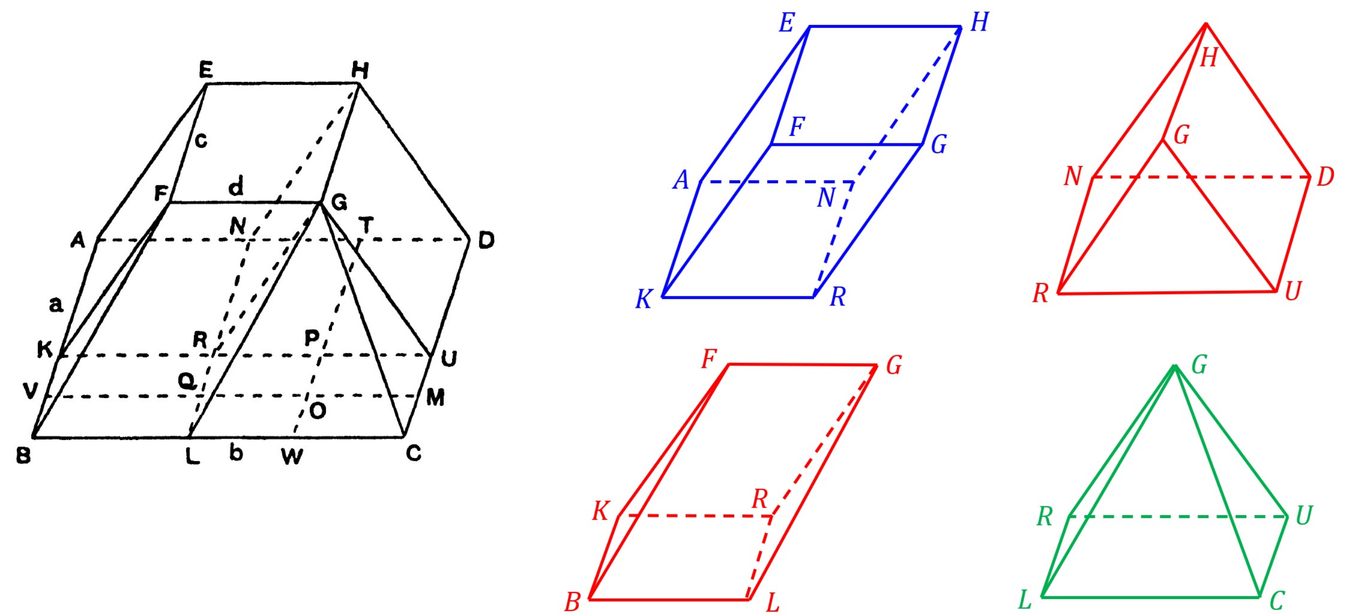
Note 6.6.E. Book I of *Metrica* gives two methods for finding the area of a triangle. The first method is based on results from Euclid's *Elements* Book II, appears in Chapter 4 of Heron's Book I, and gives the area as $\frac{1}{2}$ of the base times height of the triangle. The second method is in Chapter 8 and is known as "Heron's Formula." This states that triangle ABC with sides of lengths a, b, c (so that the perimeter is the perimeter is $a+b+c$, which we set equal to $2s$) has area $\Delta = \sqrt{s(s-a)(s-b)(s-c)}$. That is, the area is given in terms of the lengths of the sides of the triangle. A proof of Heron's Formula is to be given in Eves' Problem Study 6-11(d). The steps of the proof are given, and justification is to be provided. In the setting of the Pythagorean Theorem, Book I Chapter 9 includes a method of approximation of square roots. If A is a non-square number and a^2 is the nearest square (whole) number to it (so that $A = a^2 + b$ or $A = a^2 - b$; we could also take a as a fraction that produces a value a^2 close to A) then Heron takes as a first approximation to $\sqrt{A}$ the value $a_1 = \dfrac{1 + A/a}{2}$. A second approximation is $a_2 = \dfrac{1 + A/a_1}{2}$, and the process can be iterated from there. In Eves' Problem Study 6.11(f), Heron's method is to be used to approximate $\sqrt{3}$. If we take $a_1 = 5/3$

then we have $a_1^2 = (5/3)^2 = 25/9 = 2\frac{7}{9}$, and we find that $a_2 = 1351/780$; in fact, $(1351/780)^2 \approx 3.0000016$. This is the upper bound on $\sqrt{3}$ used by Archimedes in his approximation of π as $3\frac{10}{71} < \pi < 3\frac{1}{7}$ in his *Measurement of a Circle*; see Note 4.8.B in Section 4.8. A Chronology of π. WOW! In Chapters 11–16, areas of quadrilaterals are given. In all cases here, the measurements reduce to that of rectangles and triangles. In Chapter 17 it is shown for an equilateral triangle with base a and altitude p, that $a^2 : p^2 = 4 : 3$. From this he derives that an equilateral triangle has area $\sqrt{3}a^2/4$, where a is the length of a side (as we well know, from the Pythagorean Theorem). Heron is interested in useful numerical values, and tends to approximate "awkward" quantities like $\sqrt{3}$ and π in his presentations. He illustrates many of his formulas with specific numbers, in which case he will approximate such quantities. In Chapter 18, the area of a regular pentagon is considered. The actual formula for the area of a regular pentagon with side of length a is $A = \frac{1}{4}\sqrt{5(5+2\sqrt{5})}a^2$. So this involves $\sqrt{5}$ which, in some examples, he estimates as $9/4$ (we have $(9/4)^2 = 5.0625$). Chapter 19 covers the regular hexagon. With a side of length a, a regular hexagon simply consists of 6 equilateral triangles with sides of length a. Since the the area of such a triangle is $\sqrt{3}a^2/4$, then the area of the regular hexagon is $A = 6\sqrt{3}a^2/4 = 3\sqrt{3}a^2/2$, or as Heron expressed it, $A^2 = 27a^2/4$. In Chapter 20, Heron explores a heptagon (i.e., 7-sided polygon) inscribed in a circle of radius r. For the length of the side of the heptagon, he takes the apothem (that is, a line segment from the center of the circle that intersects a side of the polygon at a right angle) of a regular hexagon inscribed in the same circle. From properties of an equilateral triangle, is is know that the apothem of such a hexagon is $\sqrt{3}r/2$, so he is approximating the length of the side of the heptgon with $a = \sqrt{3}r/2$ (or, using

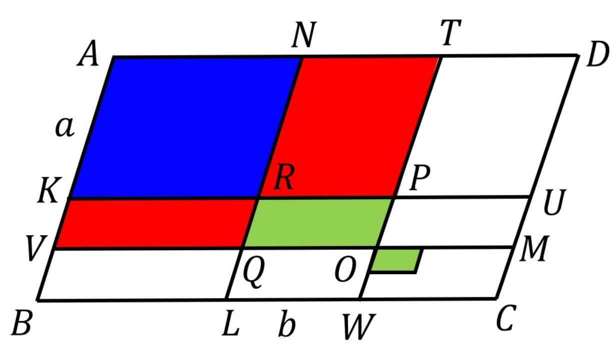
7/8 as an approximation of $\sqrt{3}/2$, $a = 7a/8$). Heron then (ultimately) estimates the area of the heptagon as $\frac{43}{12}a^2$. In Eves' Study Problem 6.11(a), the accuracy of this idea is addressed. In Chapters 21, 23, 25 of Book I, Heron finds formulas for the area of a regular octagon, decagon, and dodecagon. These involve $\sqrt{2}$ and $\sqrt{3}$, which he approximates and gives formulas $A_8 = \frac{29}{6}a^2$, $A_{10} = \frac{15}{2}a^2$, and $A_{12} = \frac{45}{4}a^2$, respectively. Regular 9-gons and 11-gons (i.e., "enneagons" and "hendecagons") are addressed in Chapter 22 and 24. This requires the use of a Table of Chords (and hence approximations), "presumably Hipparchus's Table" (according to Heath's *History, Volume 2*, page 329); see Note 6.5.C of Section 6.5. Hipparchus, Menelaus, Ptolemy, and Greek Trigonometry. Chapter 26 concerns the area of the circle, and Heron uses Archimedes approximation of 22/7 for π. In Chapters 27–32, the area of a segment of the circle is addressed. Following Archimedes' technique on the quadrature of the parabola (see Note AW.C of Supplement. The Content of Archimedes' Work, Part 1 and notice the method of exhaustion argument). However, as usual, Heron give approximations. Book I of *Metrica* concludes with Chapters 34–39 which cover the areas of an ellipse, a parabolic segment, the surface areas of a cylinder and right cone, surface area of a sphere and a segment of a sphere. In the last two cases, he borrows from Archimedes. For surface areas of a cylinder and right cone, he "unrolls" the surface giving a flat region that is a parallelogram or a sector of a circle, respectively.

Note 6.6.F. Book II addressed volumes of solids. Chapters 1–7 deal with volumes of a cone, cylinder, parallelepiped, triangular prism, pyramid with any base, and a frustum of a triangular pyramid. In each case, the solids are allowed to be oblique.

Chapter 8 considers a rectilinear solid with base a rectangle $ABCD$, side opposite the base a rectangle $EFGH$ where the sides of $EFGH$ are, respectively, parallel to the sides of rectangle $ABCD$, the sides of the rectangles need not be proportional to each other. The lateral surfaces are trapezoids joining the corresponding parallel sides of the upper and lower triangle. See the figure below. Such a solid is an example of a prismatoid. Heron finds the volume of the solid by partitioning it into a parallelepiped, two prisms, and a pyramid; a parallelepiped is produced with the same volume as Heron's solid. Introduce point K such that AK equals EF (in this derivation, it is assumed that $AB \geq EF$) and point L such that BL equals FG (again, is is assumed that $BC \geq FG$). Bisect BK and CL at points V and W, then draw $KRPU$ and $VQOM$ parallel to AD. Draw $LQRN$ and $WOPT$ parallel to AB. Introduce line segments FK, GR, LG, GU, and HN. These are used to partition the solid and create a new parallelepiped. In the figure, the parallelepiped is given in blue, the prisms are given in red, and the pyramid is given in green.



We build a parallelepiped from these which has the same volume as Heron's solid. (1) With h as the altitude of the solid, the parallelepiped has volume equal to the area of the base $AKRN$ and height h. The base has area $(AK)(KR)$. (2) The volume of the prism with lower face $KBLR$ is the same as half of the parallelepiped with this lower base and the upper base of the same size and shape (recall that $FG = BL$). We use $KVQR$ as the lower base of such a parallelepiped. (3) The volume of the prism with lower face $NRUD$ is similarly equal to half the parallelepiped with this lower base and the upper base the same size and shape. We use $NRPT$ and the lower base of such a parallelepiped. (4) The pyramid in green has volume 1/3 the area of the base times the altitude; this is 1/3 the volume of the parallelepiped with the same base and upper base of the same size and shape. The rectangle $RQOP$ contains 1/4 of the area of the base. To this, we add three more copies of $RQOP$ to get a total area of the base of $4(RQ)(RP)$. Then the volume of the parallelepiped with lower base $RLCU$ and the upper base of the same size and shape is $4(RQ)(RP)h$. The volume of the pyramid is then $(4/3)(RQ)(RP) = (RQ)(RP) + (1/3)(RQ)(RP)$. So if we create a parallelepiped of base equal to $(AV)(AT) + (1/3)(RQ)(RP)$ and altitude h, this parallelepiped will have volume equal to Heron's solid.



We have $a = AB$, $b = BC$, $c = EF$, and $d = FG$ (so that the base rectangle has sides of lengths a and b, and the upper rectangle has sides of lengths c and d). From this, $RQ = \frac{1}{2}(a-c)$, $RP = \frac{1}{2}(b-d)$, $AV = (AB) - (VB) = a - (RQ) = a - \frac{1}{2}(a-c) = \frac{1}{2}(a+c)$, and $AT = (AD) - (TD) = b - (RP) = b - \frac{1}{2}(b-d) = \frac{1}{2}(b+d)$. The volume is then:

$$\begin{aligned} V &= \left((AV)(AT) + \frac{1}{3}(RQ)(RP)\right) h \\ &= \left(\left(\frac{1}{2}(a+c)\right)\left(\frac{1}{2}(b+d\right) + \frac{1}{3}\left(\frac{1}{2}(a-c)\right)\left(\frac{1}{2}(b-d)\right)\right) h \\ &= \left(\frac{1}{4}(a+c)(b+d) + \frac{1}{12}(a-c)(b-d)\right). \end{aligned}$$

In Eves' Problem Study 6.11(g) a general prismatoid is defined which is similar to the solid considered by Heron. In the general case, the upper and lower surfaces are allowed to be any polygon, with vertices of the upper joined to the lower to form a rectilinear solid with lateral surfaces that are either triangles or trapezoids. It is to be shown in 6.11(g) that Heron's formula is equivalent to the volume formula for a prismatoid given in 6.11(g). In Chapters 9 and 10 the volume of the frustum of a cone is given (that is, a cone with the upper pointed part cut off) is given, computed based on volumes of cones in terms of circumscribing pyramids, and as a difference of two cones. Again, Heron approximates π with 22/7. In Chapter 11 he considers volumes of sphere, and in Chapter 12 the volume of a segment of a sphere; he bases both on work of Archimedes. Chapter 13 considers the volume of a torus. Heron uses a formula from Dionysodorus (circa 250 sc bce–circa 190 BCE) which he illustrates with specific numbers and approximations. Chapters 14 and 15 cover solids based on Archimedes' *Method*. Chapters 16, 17, and 18 give the volumes of he regular solids tetrahedron, octahedron, icosaheron, and dodecahedron (the cube being obvious).

Note 6.6.G. Book III of *Metrica* concerns divisions of figures. The most common agenda is to divide a plane figure by a line, or divide a solid by a plane in such a way that the resulting two parts are in a given ratio. Chapters 1–3 involve dividing a triangle with a line (1) through a vertex, (2) parallel to a side, and (3) through any point on a side. Chapters 5–8 consider a similar problem as applied to a trapezoid. Chapter 9 considers dividing a circle into a given ration using smaller circle with the same center. Chapters 11–13 consider problems similar to those concerning triangles and trapezoids, but for general quadrilaterals. The same problems are solved for any polygon in Chapters 14 and 15. Chapter 17 relates to cutting a sphere with a plane such that the resulting surface areas are in a given ratio. Chapter 18 concerns cutting a circle into three equal (area) parts with two lines. The last problem (in Chapter 20) concerning division of an area, is to find a single point in a triangle that, when joined to each of the vertices of the triangle, results in three sub-triangles of equal area. Moving on to solid figures, Chapters 20, 21, and 22 concern dividing the solid into two parts in a given ratio. These chapters cover a pyramid with any base, a cone, and a frustum of a cone, respectively. Each of these require the cube root of a number that is not, in general, a perfect cube. Consider, for example, the problem of cutting off the top of a cone of altitude h and base radius r in such a way that the top is to the rest of the cone as m to n. The volume of the whole cone is $\frac{1}{3}\pi r^2 h$. We want the fractional part $m/(m+n)$ of the volume of the whole cone to equal the volume of the upper ("pointed") part. Let r' and h' be the (unknown) radius and altitude of the top part. The top part of the cone has the same ratio of radius to altitude as the whole cone, so $r/h = r'/h'$.

Now we want

$$\frac{m}{m+n}\frac{1}{3}\pi r^2 h = \frac{1}{3}\pi(r')^2 h' = \frac{1}{3}\pi\left(\frac{rh'}{h}\right)^2 h' \text{ or } \frac{m}{m+n} = \left(\frac{h'}{h}\right)^3.$$

So in terms of h, the desired value of h' is then $h\sqrt[3]{m/(m+n)}$ and a cube root is needed. We now consider Heron's approximation of cube roots. In a numerical example of the cone problem, Heron computes a cube root of 100 as $4\frac{9}{14}$. In fact $\left(4\frac{9}{14}\right)^3 \approx 100.0820$. G. Wertheim proposes the following algorithm that Heron used. Let A be a given (positive) non-cube. For a with $a^3 < A < (a+1)^3$, let $d_1 = A - a^3$ and $d_2 = (a+1)^3 - A$. Then an approximation of $\sqrt[3]{A}$ is

$$\sqrt[3]{A} \approx a + \frac{(a+1)d_1}{(a+1)d_1 + ad_2}.$$

Wertheim's results appear in "Herons Ausziehung der irrationalen Kubikwurzeln [Heron's extraction of the irrational cube roots]," *Zeitschrift für Mathematik und Physik* [Journal for Mathematics and Physics], **44** (1899), Historisch-litterarische Abteilun, 1–3. This is available online (in German) on the Göttingen Digitization Center (go to page 395 in the pull-down "pages" menu to see Wertheim's work; accessed 5/13/2024). With $a = 4$ we have $a^3 = 4^3 = 64 < 100 < 125 = 5^3 = (a+1)^3$, as needed. The $d_1 = 100 - 64 = 36$ and $d_2 = 125 - 100 = 25$. Heron would then get

$$\sqrt[3]{100} \approx a + \frac{(a+1)d_1}{(a+1)d_1 + ad_2} = 4 + \frac{(5)(36)}{(5)(36) + (4)(25)} = 4 + \frac{180}{180 + 100} = 4\frac{9}{14}.$$

This formula can be derived from elementary algebraic manipulations, that would have been known in Heron's time. This was shown by Gustav Eneström in "Kleine Mitteilungen [Small Messages]," *Bibliotheca Mathematica*, **8** (1907–08), 412–413. This is available online (also in German) on the Internet Archive website (accessed

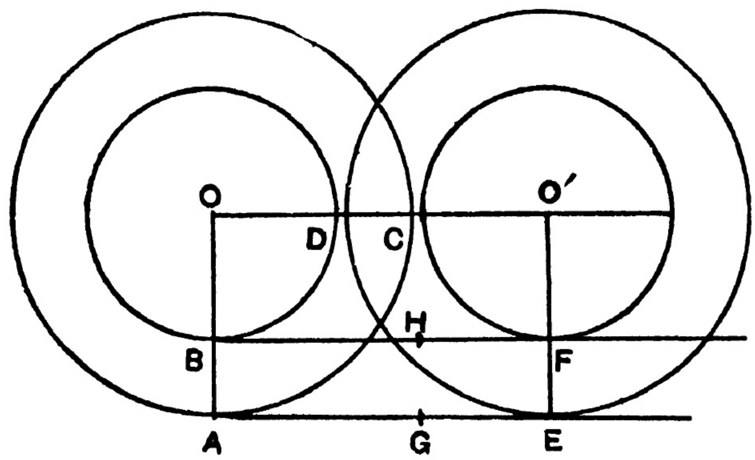
5/13/2024). Heath give an explanation of the elementary derivation of the formula in English in *History, Volume 2*, pages 341–342. In Chapter 22, a similar problem is considered, but it is the frustum of a cone (instead of a cone itself) that is cut to produce two parts with volumes in a given ratio. Heath says that, with this result, we "shall end our account of the *Metrica*" (page 342). It is unclear if this is the last result in *Metrica*, or simply the last result discussed by Heath.

Note 6.6.H. *Geometria* if based on Heron's work, but it has seen some edited and additions have been made. The measurements of areas involves the same figures as considered in Book I of *Metrica*, only there is no explanation of the formulas but they are illustrated with several specific examples. Chapters 1–4 are introductory and include definitions. Chapters 5–99 are related to the content of *Metrica* just mentioned. Johan Hieberg includes in his translation of *Geometria* material from *Liber Geeponicus* ("a badly ordered collection consisting to a large extent of extracts from the other [Heron] works," Heath *History, Volume 2*, page 318). This material includes definitions from *Definitiones*. Heiberg also adds eleven sections from the Constantinople manuscript of *Metrica*, mostly related to areas of squares, circles, segments of circles, and triangles inscribed in other such figures. Heiberg also includes material involving Heron's solving of quadratic equations; Heron completes the square in his examples. In addition, he includes some indeterminate problems; these will be discussed later in Section 6.7. Ancient Greek Algebra.

Note 6.6.I. *Stereometrica* is in two books. Chapters 1–40 of Book I consider measurements of solid figures, including the sphere, cone, frustum of a cone, obelisk with a circular base, and pyramids. Chapters 41–54 apply these results to certain buildings and other structures. Book II contains similar material, and includes the use of lengths of shadows to find heights of objects (similar to some of the work of Thales). *Geodaesia* contains extracts from *Geometria*, with the first 16 chapters of *Geodaesia* containing the material of Chapters 5–31 of *Geometria*. Chapters 17–19 give Heron's Formula, thus reproducing material from Book I of *Metrica* (Chapters 5–8; see Note 6.6.E.). *Mensurae* is attributed to Heron in an Archimedes manuscript dating to the 9th century, but the extant manuscript cannot be Heron's original version. Sections 2–27 measure a large variety of real-world objects (eg., a pillar, a tower, and a vault). Chapters 28–48 measure geometric figures and segments of them, repeating material from *Metrica* Books I and II, and *Steroemetrica*. Sections 49–59 concern measurements of other plane and solid figures.

Note 6.6.J. The *Dioptra* includes "heights and distances" problems, areas of plane figures, and some measurements related to solids. However, the mission of this work is to solve applied engineering problems. The first five chapters describe the instrument called a *dipotra*, which is an ancient theodolite (a device used in surveying to measure angles). Some of the height and distance problems are: Determine the difference of level between two given points (Chapter 6), draw a straight line connecting two points the one of which is not visible from the other (Chapter 7), the distance of two inaccessible points (Chapter 9), and the height of an inaccessible point (Chapter 12). Problems that specifically might be categorized as "engineering" (though they are more closely related to surveying) are: Find the depth of a ditch (Chapter 13), to bore a tunnel through a mountain going straight through from one given point to another (Chapter 15), and to construct a ceiling that has a surface of a given segment of a sphere. Measurement problems include dividing a given area into two parts by a straight line that produces parts with areas in a given ratio, and another statement of Heron's Formula for the area of a triangle in terms of the lengths of the three sides (see Note 6.6.E). Chapter 34 (there seem to be 38 chapters total) describes a device referred to as a "hodometer," which is an arrangement of gears and rotating screws, and is designed to measure distance. In Chapter 35 it is shown how to find the distance between Rome and Alexandria along a great circle by observing the same eclipse at these two locations. For this, Eratosthenes' estimate of the Earth's circumference from his *On the Measurement of the Earth* is used (see Section 6.3. Eratosthenes) and it is stated that Eratosthenes' estimate is the most accurate of that time. It is speculated that Chapters 34 and 35 (along with some other of the last few chapters) were added over the years by editors trying to make the work more complete.

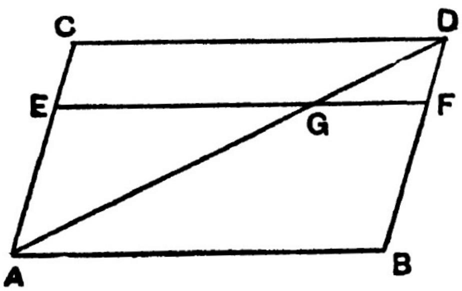
Note 6.6.K. *Mechanics* in three books survives in fragments. It survives in Arabic and seems to be heavily edited from its original form. Book I starts with the problem of moving a given weight with a given force by transferring the force through a sequence of gears. Chapters 2–7 consider motion of wheels moving on different axes. For example, Chapter 7 discusses Aristotle's Wheel.



Consider two circles with the same center, one of radius (OB) and the larger one of radius (OA). Let AC and BD be 1/4 of the circumference of each respective circle. Suppose the larger circle rolls along line AE until OCcoincides with $O'E$. At this time, point D on the smaller circle will be at point F. In this movement, arc AC is mapped to segment AE while arc BD is mapped to BF. Now $(BF) = (AE)$ so the "paradox" is how to explain that, in the rotation, the smaller wheel and the larger wheel can trace out the same distance. Heath in *History, Volume 2* (page 348) states:

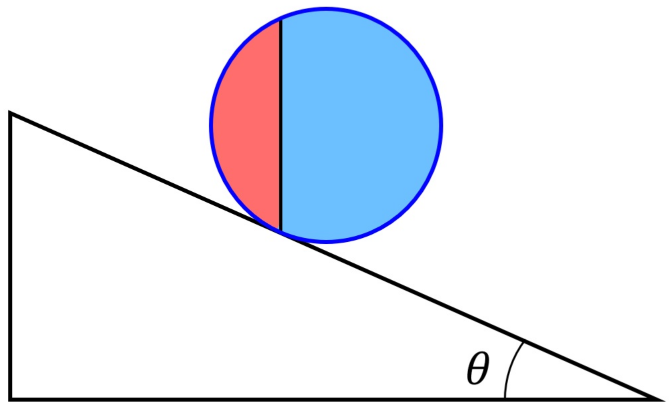
> "Heron's explanation is that, e.g. in the case where the larger circle rolls on AE, the lesser circle maintains the same speed as the greater because it has *two* motions; for if we regard the smaller circle as merely fastened to the larger, and not rolling at all, its centre O will move to O' traversing a distance OO' equal to AE and BF; hence the greater circle will take the lesser with it over an equal distance, the rolling of the lesser circle having no effect upon this."

The quicker explanation is that the smaller circle *slips* on BF as the larger circle rolls over AE. An animation illustrating the slipping of the smaller circle is given on the Wikipedia page on Aristotle's wheel paradox (accessed 5/15/2024). In Chapter 8, Heron gives the parallelogram law of addition of velocity vectors. You see this for vectors in $\mathbb{R}^n$ in Linear Algebra (MATH 2010) in Section 1.1. Vectors in Euclidean Space.



Suppose a point moves with uniform velocity along a straight line AB from A to B. At the same time, AB moves with uniform velocity always remaining horizontal with point A moving along line AC and point B moving along line BD. Suppose that the point arrives at B when the line AB reaches line CD. For EF any intermediate position of AB, let G the position of the point when AB coincides with EF. The uniform motions imply that $AE : AC = EG : EF$ and hence $AE : EG = AC : EF = AC : CD$. So G lies on the diagonal AD and this represents the path of the moving point. Chapters 9–19 concern the construction of plane and solid figures which are similar to given figures and, in terms of area and volume respectively, in a given ratio. Heron accurately saw the addition of vector quantities (well, as least velocity vectors) in his Chapter 8. In Chapter 23 he considers another physics problem: motion on an inclined plane. It is actually force instead of motion that he considers, which he introduces by considering the force that would be needed to hold a cylinder of a given weight stationary when placed on an incline plane. See the figure below which gives a cross section of the physical situation. This is covered in a standard introduction to physics class.

For example, see my online notes for Technical Physics 1 (PHYS 2110) (a work in progress, as of summer 2024) on Chapter 5, "Force and Motion—I."



Heron argues that the vertical plane passing through the line of contact of the cylinder with the inclined plane partitions the cylinder into two parts (represented in red and blue in the figure). He claims that the force along the incline plane that will hold the cylinder in place is the *difference* of the larger downhill part (in blue) and the smaller uphill part (in red). It seems that he thinks of the red part as "wanting" to fall uphill and the blue part "wanting" to fall downhill; this he takes a difference. In Exercise 6.6.A it to be shown that the force given by Heron's approach is $2(\sin\theta\cos\theta+\theta)/\pi$ times the weight of the cylinder, where θ (in radians) is the angle of inclination of the inclined plane, as indicated in the figure. This is not accurate, since we know from elementary physics that the force applied along the inclined plane that will hold a cylinder of weight w in place is $w\sin\theta$. So Heron had *some* accurate ideas about vectors (e.g., adding velocity vectors), but he did not have a modern understanding of forces and force vectors. Chapter 24 of *Mechanics* Book I concerns the center of gravity, and Chapter 25–31 cover supporting a heavy beam or a wall by a number of pillars. Chapters 32 and 33 concern levers, and Heron credits Archimedes as giving authoritative foundational material.

Note. 6.6.L. Book II of *Mechanics* is related to mechanical "powers" (that is, machines that allow a small force to be applied to a heavy weight). Heron includes wheels and axles, levers, pulleys, the wedge, the screw, and combinations of these. The first six chapters give descriptions of the machines. These are followed by several chapters giving specific illustration of the use of the machines. Chapter 32 discusses the effect of friction on the machines power. Chapter 34 discusses questions raised by Aristotle. Examples of such questions are: (1) "Why do great weights fall to the ground in a shorter time than lighter ones?" and (2) "Why does a stick break sooner when one puts one's knee against it in the middle?" Of course Aristotelean physics was not entirely correct, and the assumption that heavy objects fall faster than light ones in (1) is false (it is a question of air resistance). Heron returns to centers of gravity in Chapters 35, 36, and 37 and finds the center of gravity of triangles, quadrilaterals, and pentagons. In the remaining four chapters of Book II, centers of mass of such figures (and polygons in general) are considered when additional weights are put on the angles of the figures. Book III deals with the actual construction of machines which use pulleys (Heath, *History, Volume 2* only devotes one sentence to Book III).

Note. 6.6.M. Heron's work *Catoptrica* contains several theoretical propositions that are also in Euclid's work titled *Catoptrica* (or *Optics*, as it was called in Section 5.2. Euclid and Section 5.8. Euclid's Other Works; see Note 5.8.D). Heron's work also addresses the problem of configuring mirrors in a certain way so that objects are reflected in a certain way. For example, the construction of a system of mirrors that makes the right side appear on the right, instead of reversed; in this way, a written page would be reflected in such a way that it appears in its original orientation. Heron includes concave and cylindrical mirrors in his collection of arrangements of mirrors. All of this work is based on the property that the angle in incidence equals the angle of reflection for any reflecting surface. This is shown in Propositions 4 and 5 of *Catoptrica* and is also to be shown in Eves' Problem Study 6.11(b) based on the fact that light follows the shortest path between its source and the point where it is ultimately detected (i.e., the eye that sees it). See the figure below, in which the claim is that $\alpha = \beta$.