6.8. Diophantus

Note. Like Heron, the dates of Diophantus are uncertain. The MacTutor biography page (accessed 5/17/2024) gives the birth and death years of circa 200 CE and circa 284 CE. In this section, we concentrate on his two partially surviving works, *Arithmetica* and *On Polygonal Numbers*. *Arithmetica* plays a central role in the development of algebra and number theory, including the introduction of symbols to mathematics (and the beginning of the Syncopated Algebra stage; see Note 6.7.A of Section 6.7. Ancient Greek Algebra).

An imaged appearance of Diophantus, from the Greatest Greeks webpage on Diophantus (accessed 5/17/2024)

Note 6.8.A. Diophantus' age is the topic of one of the problems in the *Greek Anthology* (for information on the *Greek Anthology*, see Note 6.7.B of Section 6.7. Ancient Greek Algebra). This problem is given by Eves as Problem Study 6.15(a) (the solution gives an age of 84 when Diophantus died). The known works of Diophantus are the *Arithmetica*, *On Polygonal Numbers*, and possibly *Porisms* (or "Corollaries"; this would include results implies by previously known results). *Arithmetica* was originally in thirteen books, but only six of the books survive.

We'll explore its content in detail below. Only a fragment of *On Polygonal Lines* survives. In Book V of *Arithmetica*, Diophantus refers to "the *Porisms*" in his proofs of Propositions 3, 5, and 16. It seems that this could be a separate collection of propositions that has not survived. We now turn our attention to *Arithmetica*.

Note 6.8.B. Johann Regiomontanus (June 6, 1436–July 6, 1476) was the first of the Renaissance-era to call attention to the work of Diophantus by observing that there was no translation from Greek into Latin of "Diofantus, a Greek arithmetician." Rafael Bombelli (January 1526–1572) was the first to find a manuscript of *Arithmetica* in the Vatican. Around 1570, he and Antonio Maria Pazzi translated five books out of the seven, but did not publish this translation. However, Bombelli took all of the problems from the first four Books ans some of the fifth and published them, along with some of his own problems, in his 1572 *Algebra*. In 1575, Wilhelm Holzmann (commonly known as "Xylander"; December 26, 1532–February 10, 1576) produced a carefully-prepared Latin translation of Diophantus, along with commentary. For some 200-odd years, the standard edition of Diophantus was Claude Gaspar Bachet's (October 9, 1581–February 26, 1638) version which included the Greek text along with the Latin translation and supplemental notes in 1621. A second edition was printed in 1670 which, unfortunately, was carelessly printed and contained errors in the test, but which contained notes made by Pierre de Fermat (August 17, 1601–January 12, 1665). In one of these notes, Fermat wrote: "It is impossible for a cube to be written as a sum of two cubes or a fourth power to be written as the sum of two fourth powers or, in general, for any number which is a power greater than the second to be written as a sum of two like powers. I have a truly marvelous demonstration of this proposition which this margin is too narrow to contain." That is, there are no natural number solutions a, b, c to the equation $a^n + b^n = c^n$ where n is a natural number (i.e., positive integer), except in the case where $n = 2$ (in this case, the solutions form Pythagorean triples). This unproved conjecture stood for about 350 years, until it was proved true in the 1990s. More information on this can be found in Supplement. Fermat's Last Theorem—History, a supplement to our Section 10.3, "Fermat." Simon Stevin (1548–February 1620) published a French version of the first four Books in 1585, based on Xylander's earlier work. Albert Girard (1595–December 8, 1632) added Books V and VI to this and published the complete edition in 1625. Several other translations based on Bachet's version include:

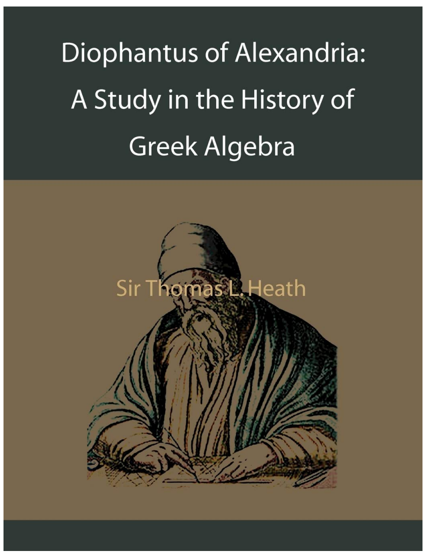
1. Otto Schulz gave a German translation in 1822.

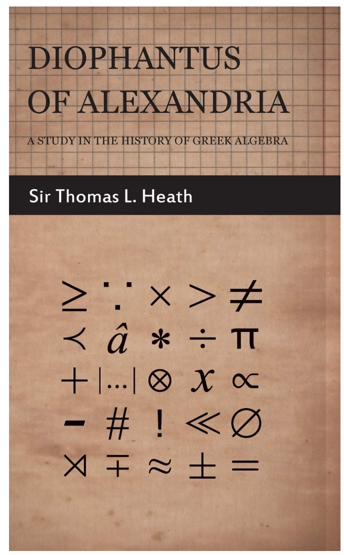
2. Gustav Wertheim (June 9, 1843–August 31, 1902) published another German version in 1890.

3. Friedrich Poselger (1771–1838) translated a fragment of *Polygonal Numbers* in 1810.

Another German version is that of Paul Tannery, *Diophantus alexandrus. Opera omnia* [Diophantus of Alexandria, Complete Works], 2 volumes, Liepzig: B.G. Teubner (1893, 1895). This version is "much superior to that of Bachet" (as Heath states on page v of his translation of this). This is the version referenced by Thomas Heath in his work (in English) *Diophantus of Alexandria: A Study in the History of Greek Algebra*, 2nd Edition, Cambridge: Cambridge University Press (1910). This note is based on Heath's *History, Volume 2*, pages 453–455.

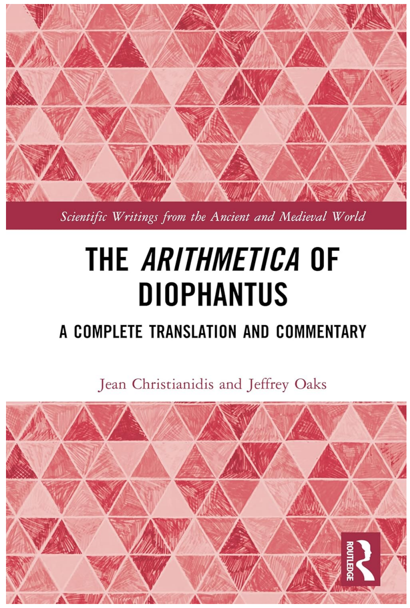
Note 6.8.C. Perhaps unsurprisingly, Diophantus' *Arithmetica* is another work translated into English by Thomas Heath (see Note 5.3.K. of Section 5.3. Euclid's "Elements" for more on him). He published *Diophantus of Alexandria: A Study in the History of Greek Algebra*, 2nd Edition, Cambridge: Cambridge University Press (1910). It is still in print by Martino Publishing (2009) and Heath Press (2008).





In print versions of Heath's work on Diophantus,
from Amazon.com (accessed 5/17/2024)

A PDF copy is available online on Archive.org (accessed 5/17/2024). There is a new translation of *Arithmetica* by Jean Christianidis and Jeffrey Oaks: *The Arithmetica of Diophantus: A Complete Translation and Commentary*, (Routledge, 2022); see the figure below. Hypatia of Alexandria (circa 370 CE–March 415 CE) wrote the first commentary on *Arithmetica* Books I–VI, but wrote noting on the remaining books. Arabic commentaries and similar studies (some of which reproduce problems from *Arithmetica*) dating around 1000 CE make no mention of the other books, implying that Books VII–XIII were lost before the 10th century.



A new translation of *Arithmetica*, from Amazon.com (accessed 5/17/2024)

These commentaries include ones by (using the spelling and dates given in Heath's *History, Volume 2*, page 453) Abū'l Wafā al-Būsjānī (940–998), Qusttạ̄ al-Baʻlabakkī (circa 900), and Ibn al-Haitham (circa 965–1039). We saw Abū'l Wafā al-Būsjānī and Ibn al-Haitham in Section 5.3. Euclid's "Elements" in reference to translations of the *Elements*.

Note 6.8.D. Our word "arithmetic," as well as the title of Diophantus work, comes from the Greek words *arithmos* (meaning "number") and *techne* (meaning "science"). Diophantus introduces the beginnings of symbolic algebraic notation, usually by taking the first two letters of a relevant word. The symbol used for an unknown is a lowercase sigma, ς (In LaTeX, this is "`\varsigma`"). Heath argues (in *History, Volume 2*, page 457) that this symbol evolved from the contraction of the first two letters of *arithmos* (or $\alpha\rho\iota\theta\mu o\varsigma$), alpha and rho. Powers of the unknown are represented similarly. The "unknown squared" is denoted Δ^{Υ}, derived from *dunamis* (or $\Delta\Upsilon\mathrm{NAMI}\Sigma$ for "power"). The "unknown cubed" is denoted K^{Υ}, derived from *kubos* (or $K\Upsilon BO\Sigma$ for "cube"). The "unknown square-square" is denoted $\Delta^{\Upsilon}\Delta$, the "unknown square-cube" is denoted ΔK^{Υ}, and the "unknown cube-cube" is denoted $K^{\Upsilon}K$. The following table summarizes this.

Variable	**Symbol**	**Greek Letters**
unknown, x	ς	sigma
unknown squared, x^2	Δ^{Υ}	Delta Upsilon
unknown cubed, x^3	K^{Υ}	Kappa Upsilon
unknown square-square, x^4	$\Delta^{\Upsilon}\Delta$	Delta Upsilon Delta
unknown square-cube, x^5	ΔK^{Υ}	Delta Kappa Upsilon
unknown cube-cube, x^6	$K^{\Upsilon}K$	Kappa Upsilon Kappa

The symbol for subtraction is ⋔. This is thought of an inverted capital lambda, Λ, with a superimposed capital iota, I. The Greek word for "lacking" is *leipis* (or $\Lambda\mathrm{EI}\Psi\mathrm{I}\Sigma$). There is no symbol for addition; it is represented by juxtaposition. That is, "terms" that are next to each other are assumed to be added together. To add and subtract several terms, by convention, the terms added appear first (i.e., on the left), then the symbol for subtraction is given, and on the right the terms to be subtracted. For example, if we wish to represent the operations

$$51 - 1 - 28 + 624 + 71 - 537$$

as Diophantus would, then we have (see the table in Section 1.6. Ciphered Numeral Systems for the representation of numbers using the Greek alphabet)

$$\nu\alpha\chi\kappa\delta o\alpha \pitchfork \alpha\kappa\eta\phi\gamma\zeta.$$

That is,

$$51 - 1 - 28 + 624 + 71 - 537 \Longleftrightarrow \nu\alpha\chi\kappa\delta o\alpha\pitchfork\alpha\kappa\eta\phi\gamma\zeta.$$

The Greek representation is unambiguous, but difficult (I think!) to process without any spacing or punctuation. Lets decipher this by introducing some spaces and parentheses:

$$(\nu\alpha)\,(\chi\kappa\delta)\,(o\alpha)\pitchfork(\alpha)\,(\kappa\eta)\,(\phi\gamma\zeta) \Longleftrightarrow (51 + 624 + 71) - (1 + 28 + 537).$$

Of course, we could also just sum all of the added symbols (with disregard to the order of appearance) and then subtract the sum of all of the subtracted symbols. To express a polynomial, the power of the variable appears first and then the coefficient. The rules for sums and differences are otherwise as just discussed. For constant terms in a polynomial, the symbol $\mathring{\mathrm{M}}$ is placed before the constant, where this is symbol motivated by the Greek word *monades* (or ΜΟΝΑΔΕΣ for "units"). With this notation, we have the following examples of representation of polynomials:

$$x^3 + 13x^2 + 5x \Longleftrightarrow K^{\Upsilon}\alpha\Delta^{\Upsilon}\iota\gamma\varsigma\varepsilon \text{ and } x^3 - 5x^2 + 8x - 1 \Longleftrightarrow K^{\Upsilon}\alpha\varsigma\eta\pitchfork\Delta^{\Upsilon}\varepsilon\mathring{\mathrm{M}}\alpha.$$

Let's also analyze this by introducing some parentheses and spaces:

$$\begin{array}{ccccc} K^{\Upsilon}\alpha & & \Delta^{\Upsilon}\iota\gamma & & \varsigma\varepsilon \\ 1x^3 & + & 13x^2 & + & 5x \end{array} \quad \text{and} \quad \begin{array}{ccccccccccc} & K^{\Upsilon}\alpha & & \varsigma\eta & & \pitchfork & & \Delta^{\Upsilon}\varepsilon & & \mathring{\mathrm{M}}\alpha & \\ (& 1x^3 & + & 8x &) & - & (& 5x^2 & + & 1 &) \end{array}$$

In Rhetorical Algebra, these would be literally

unknown cubed 1, unknown squared 13, unknown 5

and

(unknown cubed 1, unknown 8) minus (unknown squared 5, units 1).

So Diophantus has introduced the Syncopated Algebra stage (see Section 6.7. Ancient Greek Algebra for more on this terminology). This note is primarily based on Eves text book, pages 181 and 182 (though the polynomial examples also appear in Heath's *History, Volume 2*, pages 458–460).

Note 6.8.E. Diophantus does not present a symbol for division, but he does present one for reciprocal. With the unknown as ς, he represents the reciprocal of the unknown as ς^{χ} We then predictably have:

Variable	**Symbol**
reciprocal of the unknown, $1/x$	ς
reciprocal of the unknown squared, $1/x^2$	$\Delta^{\Upsilon\chi}$
reciprocal of the unknown cubed, $1/x^3$	$K^{\Upsilon\chi}$
reciprocal of the unknown square-square, $1/x^4$	$\Delta^{\Upsilon}\Delta^{\chi}$
reciprocal of the unknown square-cube, $1/x^5$	$\Delta K^{\Upsilon\chi}$
reciprocal of the unknown cube-cube, $1/x^6$	$K^{\Upsilon}K^{\chi}$

As before, coefficients come after the unknowns, so that

$$250/x^2 \Longleftrightarrow \Delta^{\Upsilon\chi}\sigma\nu.$$

As opposed to having a symbol for division, Diophantus writes the numerator, followed by *ἐν μορίῳ* or *μορίυ*, meaning "divided by," according to Heath's Index of Greek Words in *History, Volume 2*. We now have, for example, the rational expression

$$\frac{60x^2+520}{x^4+900-60x^2} \Longleftrightarrow \Delta^{\Upsilon}\xi\ \mathring{\mathrm{M}}\phi\kappa\ \ \acute{\epsilon}\nu\ \ \mu o\rho\acute{\iota}\omega\ \ \Delta^{\Upsilon}\Delta\alpha\ \mathring{\mathrm{M}}\text{ϡ}\ \pitchfork\ \Delta^{\upsilon}\xi.$$

A shortcoming of Diophantus' notation is that it only allows the use of one unknown at a time. So a problem involving two unknowns has to somehow be manipulated to produce two equations in one unknown (an elimination of one variable for each, if possible). Diophantus deals with indeterminate equations (i.e., those without unique solutions) by choosing *some* values for all but one of the unknowns, and then solving for the final unknown. Since the choosing could be done arbitrarily, there is *some* underlying generality of the solutions produced. . . This notes is based on Heath's *History, Volume 2*, pages 458, 460, and 461.

Revised: 5/18/2024