8.3. Fibonacci and the Thirteenth Century

Note. In this section of notes, we present Eves' material on Leonardo of Pisa (or Leonardo Pisano, also known as Fibonacci, circa 1170–circa 1250) and his *Liber abaci*. Eves' presentation is brief, but a detailed supplement to this section is available on Leonardo of Pisa (Fibonacci) and the *Liber abbaci*.

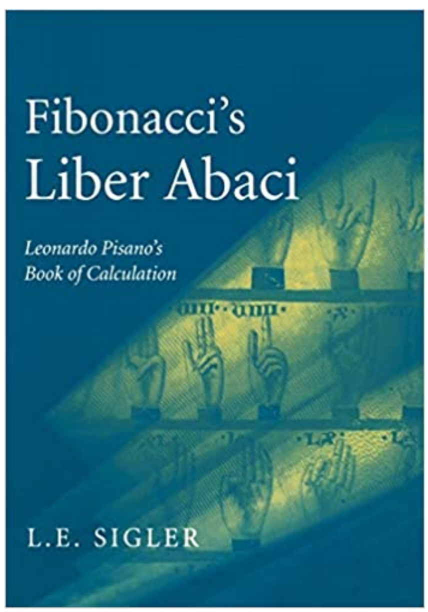
Image from the MacTutor History of Mathematics Archive biography of Fibonacci (accessed 4/8/2023)

Note. Eves' refers to Leonardo of Pisa as "the most talented mathematician of the Middle Ages" (see Eves' page 261). This may be debatable, but he certainly is one of the most influential. His book, *Liber abacci* (published in 1202 with a second revised edition published in 1228) helped spread the Hindu-Arabic numerals and their use around the Mediterranean and throughout Europe.

Note. Leonardo of Pisa's father, Guilielmo, was businessman and traveled around the Mediterranean as a representative of the merchants of the Republic of Pisa (a"city-state" at the time). Leonardo joined his father in the north African city of Bugia (today, this is the city Bejaia in Algeria). It is here where Leonardo was learned gained most of his knowledge of the Hindu-Arabic numerals and their use.

He also traveled to nearby regions (Egypt, Greece, and Syria included) where he had further exposure to the numbers.

Note. Leonardo of Pisa published the first edition of *Liber abbaci* in 1202. However, a copy of this version did not survive. He published a second edition in 1228 and copies of this version did survive. Surprisingly, *Liber abbaci* was not printed (that is, in movable type) until 1857 by Baron Baldassarre Boncompagni (in Latin with no commentary). A printed English version by Laurence Sigler was published in 2002 (based on Boncompagni's edition; notice that this occurs *after* the edition of Eves' book on which these notes are based!). Sigler's book is 672 pages in length and is the only translation into a modern language! We take Keith Devlin's *The Man of Numbers: Fibonacci's Arithmetic Revolution* (Walker and Company, 2011) page 8 as the source for these last claims (Devlin's book is also the source for the supplemental notes to this section).



Laurence Sigler's *Fibonacci's Liber Abaci: A Translation into Modern English of Leonardo Pisano's Book of Calculation*, Springer (2002)

Note. *Liber abacci* shows the influence of earlier Arabic influences, in particular the algebraic works of al-Khwārizmī (790–850) and Abuū Kāmil (850–930). The book consists of 15 chapters, the first of which describes how to read and write the whole numbers in the Hindu-Arabic notation. Chapters 2 through 7 present the techniques of addition, subtraction, multiplication, and division of the numbers, some involving fractions. The remainder of the book gives several examples and describes computation of square and cube roots, and solutions of linear and quadratic equations. Negative roots and imaginary roots are not addressed since this are not recognized as "numbers" at this point. The presentation is rhetorical and the use of variables is not addressed; symbolic algebra with variables does not appear until the work of François Viète in the 1590s (see Section 8.9. François Viète for more details). The first two paragraphs of *Liber abbaci* include much of the biographical information we have on Leonardo of Pisa.

Note. In the latter part of Chapter 12, Leonardo of Pisa writes (quoting from page 404 of Sigler's book):

> HOW MANY PAIRS OF RABBITS ARE
> CREATED BY ONE PAIR IN ONE YEAR.
>
> A certain man had one pair of rabbits together in a certain enclosed place, and one wishes to know how many are created from the pair in one year when it is the nature of them in a single month to bear another pair, and in the second month those born to bear also.

This problem introduces the *Fibonacci sequence*

$$1, 1, 2, 3, 5, 8, \ldots, x, y, x + y, \ldots.$$

Leonardo of Pisa did not invent this problem; it was already known to the Indian mathematicians (according to Devlin's *The Man of Numbers*, page 143). None-the-less, Leonardo's current fame lies with primarily with this problem. Problem 8.2 in this section deals with the Fibonacci sequence. Surprisingly, the nth Fibonacci number, F_n, is given by the formula

$$F_n = \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right).$$

This can be shown by setting up an expression for F_n using vectors in $\mathbb{R}^2$ and 2×2 matrices. The nth Fibonacci number then requires knowing the n power of the matrix. This leads to diagonalizing the matrix, which requires finding eigenvalues. The eigenvalues of the matrix are $\lambda_1 = (1 + \sqrt{5})/2$ and $\lambda_2 = (1 - \sqrt{5})/2$, so this is where the $\sqrt{5}$ enters the computations (it comes from the quadratic equation applied to the characteristic polynomial of the 2×2 matrix). For details, see my online notes for Linear Algebra (MATH 2010) on Section 5.3. Two Applications (the applications in question involve diagonalizing a matrix).

Note. *Liber abbaci* is not the only book attributed to Leonardo of Pisa. The following is from the MacTutor History of Mathematics Archive biography of Fibonacci (accessed 4/11/2023).

> Fibonacci ended his travels around the year 1200 and at that time he returned to Pisa. There he wrote a number of important texts which played an important role in reviving ancient mathematical skills and he made significant contributions of his own. Fibonacci lived in the days before printing, so his books were hand written and the only way to have a copy of one of his books was to have another hand-written copy made. Of his books we still have copies of *Liber abaci* [Book of the Abacus] (1202), *Practica geometriae* [Practical Geometry](1220), *Flos* [Flower] (1225), and *Liber quadratorum* [Book of Squares]. Given that relatively few hand-made copies would ever have been produced, we are fortunate to have access to his writing in these works. However, we know that he wrote some other texts which, unfortunately, are lost. His book on commercial arithmetic *Di minor guisa* [Book of small ways] is lost as is his commentary on Book X of Euclid's *Elements* which contained a numerical treatment of irrational numbers which Euclid had approached from a geometric point of view.

In fact, *Liber quadratorum* has also been translated by Laurence Sigler (translator of the 2002 Springer translation of *Liber abbaci*) in *The Book of Squares: An Annotated Translation into Modern English* (Academic Press, 1987). According to the MAA *Convergence* (January 2017) article "Mathematical Treasure: Fibonacci's Practica Geometriae" by Cynthia J. Huffman, Leonardo's surviving works are all contained in an 1850s Latin version (with no commentary) published in a two volumes by Baldassarre Boncompagni (we mentioned this above in connection with *Liber abbaci*). The first volume contain *Liber Abbaci* and the second volume, *Practica Geometriae ed Opuscoli* ("Practical Geometry and Lesser Works") contained

Practica geometriae the lesser works *Flos* and *Liber quadratorum* along with *Epistola ad Magistrum Theodorum*, a letter to the philosopher Theodorus). An English version of *Practica geometriae* is available in Barnabas Hughes' *Fibonacci's De practica geometrie* (Springer, 2008). Keith Devlin in *The Man of Numbers* (Walker and Company, 2011) describes *Practica geometriae* as "written for professional people whose work involved surveying and land measurement" and including "both practical instructions for performing various calculations written for artisans, as well as mathematical verifications of the methods he described for scholars" (see his page 87). Eves describes this work more concisely as giving "rigorous arguments for results in geometry and trigonometry" (see Eves' page 263). Devlin describes *Flos* as "largely devoted to algebra, and containing his solutions to a series of problems posed to him in a contest organized for Frederick [the Holy Roman Emperor Frederick II]" (see Devlin's page 88). Devlin describes *Liber quadratorum* as "a book on advanced algebra and number theory, and is Leonardo's most mathematically impressive work" (see Devlin's page 88).

Note. Problem 8.3 gives the flavor of some of the problems in *Liber abbaci*, each an elementary applied problem. They involve a systems of equations and a "rate of work" problem. Problem 8.4(a) is a problem from *Liber quadratorum*, Problem 8.4(b) is a problem from *Flos*, and Problem 8.4(c) gives another problem from *Liber abbaci*. Eves describes the following problem from *Liber quadratorum* (see page 263). Find a rational number x such that $x^2 + 5$ and $x^2 - 5$ shall each be squares of rational numbers. Leonardo gives the correct answer $x = 41/12$. This problem (and the next) are credited to John of Palermo, an advisor of Emperor Frederick

II. The next problem is to find a solution to the cubic equation $x^3+2x^2+10x=20$. The solution of cubic equations would be the motivation for much study in the 16th century, as we will see in Section 8.8. Cubic and Quartic Equations. Leonardo does not precisely solve the stated cubic equation, but instead gives an approximate solution, 1.3688081075, which is correct to nine decimal places. He "attempted a proof" (as Eves says on page 263, apparently implying that he did not succeed) that no root of the equation can be expressed in for form $\sqrt{a+\sqrt{b}}$ where a and b are rational. Eves states that this would imply that roots of the cubic are not constructible numbers. This problem appears in *Flos*. We leave additional details on Leonardo of Pisa to the supplement to this section, Leonardo of Pisa (Fibonacci) and the *Liber abbaci*.

Note. Eves mentions four others who contributed to mathematical development in the thirteenth century mathematicians: Johannes de Sacrobosco, Robert Bacon, Campanus of Novara, and Jordanus Nemorarius.

Johannes de Sacrobosco (circa 1195–1256)

Robert Bacon (1214–1292)

Campanus of Novara (1220–1296)

Jordanus Nemorarius (1225–1260)

The image of Jordanus Nemorarius is from alchetron.com webpage (accessed 4/12/ 2023). The other images are from the respective biographical pages of the Mac-

Tutor History of Mathematics Archive website. We turn to these sources for brief descriptions of these individuals.

Note. Johannes de Sacrobosco, or "John of Holywood," was English and educated at Oxford. He studied in Paris and taught mathematics at the University of Paris. He promoted Arabic methods of arithmetic and algebra in his *De Algorismo*, which had 11 chapters addressing addition, subtraction, multiplication, division, square roots, and cube roots. Sacrobosco wrote *Tractatus de Sphaera* in 1220 on astronomy and Ptolemy's theory of planetary motion. He wrote *De Anni Ratione* in 1232 which dealt with the calendar and the observation that the Julian calendar was 10 days ahead of where it should be. (Corrections were made with Pope Gregory XIII's introduction of the Gregorian calendar in 1582 and the imposition that the day which would follow Thursday October 4, 1582 was Friday October 15, 1582.)

Note. Roger Bacon, too, was English and educated at Oxford. He also taught at Oxford, and spent time in Paris. Returning to Oxford in 1247, his interest turned to mathematics and 'natural philosophy' (which was a medieval version of the natural sciences, including astronomy/physics, biology/medicine, and chemistry/alchemy). According to the MacTutor History of Mathematics Archive biography of Roger Bacon (accessed 4/13/2023), his most important mathematical contribution is the application of geometry to optics. He experimented with lenses and mirrors and his ideas on this topic are given in *De mirabile potestate artis et naturae*, a letter he wrote around 1250. He joined the Franciscan Friars in Oxford, and later in Paris.

By 1267 he had put together a three volume work, *Opus maius* (Great Work), *Opis minus* (Smaller Work), and *Opus tertium* (Third Work) which was meant to show the Pope that science was important to a university curriculum and important to the Church. As a sequel to this, he started to write *Communia naturalium* (General Principles of Natural Philosophy) and the *Communia mathematica* (General Principles of Mathematical Science). This second work contains a study of quantity and a theory of proportions (only parts of it were published, and it may not have been completed). He served 12 years in prison from 1278 to 1290 in Italy when he was charged by the church with "suspected novelties" in his teaching. He lived two years after his release from prison and his final writings were on his theological beliefs and appeared in *Compendium studii theologiae*.

Note. Jordanus Nemorarius, or Jordanus de Nemore, was born in Borgentreich (near Warburg, in modern-day Germany). He wrote six mathematical works. In his *Demonstratio de algorismo* (A Demonstration of Notation) he gives a practical explanation of the Arabic number system and considers only integers. In *Demonstratio de minutiis* (A Demonstration of Trivia), he deals with fractions. His *De elementis arithmeticae artis* (Elements of Arithmetic Arts) is a theoretical consideration of arithmetic which became a standard reference for other Middle Age texts. The topic of his *Liber phylotegni de triangulis* (Book of Study of Triangles) is geometry. His *Demonstratio de plana spera* (Demonstration of a Plane Sphere) considers geometry on a sphere and stereographic projection. His most advanced work was *De numeris datis* (On the Numbers Given), which is "the first advanced algebra to be written in Europe after Diophantus [circa 200 CE–circa 284 CE]" (according to the MacTutor History of Mathematics Archive biography of Jordanus (accessed 4/14/2023). In fact, an English translation of this work is available as *Jordanus De Nemore De Numeris Datis*, translated by Barnabas Hughes (University of California Press, 1981). A through source on Jordanus and his work is given in J Høyrup, "Jordanus de Nemore, 13th Century Mathematical Innovator," *Archive for History of Exact Science*, **38** (1988), 307–363.

Note. Campanus of Novara, or Johannes Campanus, was born in Novara (in modern-day Italy). He wrote a Latin edition of Euclid's *Elements* around 1260, and it was the standard edition of the *Elements* for 200 years. His version relied on Adelard of Bath's (1075–1160) Latin translation of Euclid from Arabic sources, and on Robert of Chester's collection of writing by commentators on Euclid. In his translation of Book VII, Campanus used definitions and axioms from Jordanus Nemorarius's *De elementis arithmeticae artis*. In 1485 Campanus's work became the first printed edition of Euclid's *Elements* when it appeared in Venice. According to the MacTutor History of Mathematics Archive biography of Campanus (accessed 4/14/2023), this is the first printed mathematical book of any importance. Most of Campanus's writings were on astronomy. He wrote *Theorica Planetarum* around 1262 and in it described the construction of an "equatorium." This is an instrument that mechanically calculates the locations of the planets by recreating the motion of a planet using Ptolemy's model. This is avialable today as *Campanus of Novara & Medieval Planetary Theory: "Theorica Planetarum,"* (Medieval Science Series; No. 16), translated with commentary by Fracis Benjamin, Jr. and G. J. Toomer (University of Wisconsin, 1972). Campanus also wrote *Computus maior* on solar and lunar cycle, and *Tractatus de Sphaera* which described the celestial events seen in one 24 hour rotation of the heavens and gave a detailed description the mositon of the planets in the night sky.

Note. Eves summarizes the thirteenth century as (see his pages 263 and 264):

> "In spite of the bleak picture often given of the mathematics of the thirteenth century, it was the early part of that century that saw the high point of medieval achievement in arithmetic, geometry, and algebra. ...It saw the rise of the universities at Paris, Oxford, Cambridge, Padua, and Naples. Universities later became potent factors in the development of mathematics, many mathematicians being associated with one or more such institutions."

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