8.5. The Fifteenth Century

Note. In this section we briefly describe the European Renaissance, and give the mathematical contributions of, among others, Nicholas of Cusa, Regiomontanus, and Luca Pacioli.

Note 8.5.A. The European Renaissance starts sometime in the 14th century, includes the 15th and 16th century, and ends sometime in the 17th century (there aren't specific events marking the beginning and end of the Renaissance, so these dates are fuzzy). The French word *renaissance* means "rebirth" due to the renewed interest in the arts and sciences (and mathematics!) of classical antiquity (i.e., ancient Greece and Rome). It starts in Italy (and the "Italian Renaissance"), specifically in Florence. Perspective was introduced into paintings (this was the time of Leonardo de Vinci and Michelangelo). Interest in learning based on the classical sources thrived, affecting architecture, literature, philosophy, science, and exploration. With the fall of Constantinople to the Turks in 1453, refugees flowed into Italy and brought with them works of classical Greece. These became sources of inspiration and, with a certain technological development at the same time, were widely circulated. This technological development was the single most influential one: movable type. This revolutionized the book trade and made printed material widely available (as opposed to depending on scribes to slowly copy the work). Johannes Gutenberg (circa 1400–February 3, 1468) had his printing press working in 1450. He was named "Person of the Century" for the 15th century in the December 31, 1999 issue of *Time Magazine* (Volume 154, Number 27); most of his issue is online on the *Time Magazine* website. The mathematical development of the 15th century concentrated on arithmetic, algebra, and trigonometry. This note is based on Eves' page 265 and the Wikipedia pages on the Renaissance and the Italian Renaissance (these websites were accessed 6/21/2023).

The skyline of Florence, Italy from *Piazzale Michelangelo*, with the *Cathedral of Santa Maria del Fiore* ("Cathedral of Saint Mary of the Flower") and its dome by architect Filippo Brunelleschi (1377–April 15, 1446) on the right and the Palazzo Vecchio Tower on the left. From the EncirclePhotos.com website (accessed 6/21/2023)

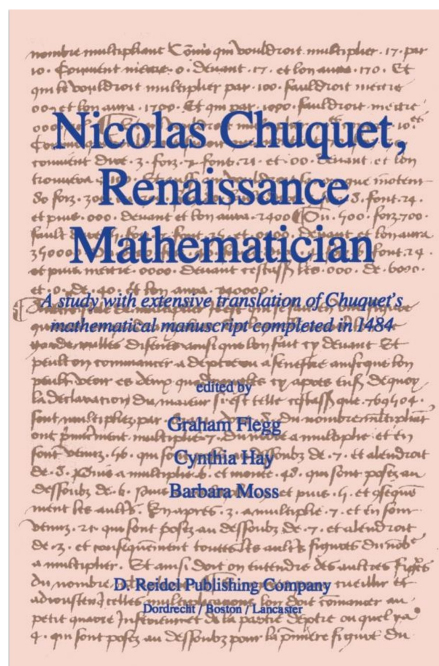
Note 8.5.B. Nicholas of Cusa (1401–August 11, 1464) was a German philosopher and bishop. He enrolled in the University of Heidelberg (Germany) in 1416 and in the University of Padua (Italy) in 1417. He studied philosophy and canon law (i.e., church law). In Padua he learned about the latest advancements in math and astronomy. He graduated from Padua with a doctorate in canon law in 1423. In 1425 he started studying divinity at University of Cologne. In the 1430s he was appointed a member of the Council of Basel which was set up to try to reform the Church. He advised the Council on calendar reform in 1436. He continued involvement with Church affairs and traveled around Germany and as far as Constantinople in his work. He was named a cardinal in 1448 as a result of his work, and became the bishop of Brixson (northern Italy) in 1450. In addition to his writings for the Church (the best summary of his religious views are in his book *De visione dei* ["The Vision of God," 1453]), his first published book was a work on philosophy, *De docta ignorantia* ("On the Doctrine of Ignorance"), which covered the incomplete nature of our knowledge of the universe. His writings on math include: *De geometricis transmutationibus* ("On Geometric Tranformations," 1445), *De arithmeticis complementis* ("On Arithmetical Complements," 1445), *De circuli quadratura* ("The Quadrature of the Circle," 1450), *De mathematicis complementis* ("On Mathematical Complements," 1453), *Dialogus de circuli quadratura* ("Dialogue on the Quadrature of the Circle," 1457), *De mathematica perfectione* ("On Mathematical Perfection," 1458), and *Declaratio rectilineationis curvae* ("Declaration on a Rectilinear Curve," unknown date). His geometric works include attempts to square the circle and trisect the general angle (Eves, page 265). Nicholas contributed to the study of the infinitely large and infinitely small. In Problem Study 8.6(c), an approximation to the circumference of a given circle is considered which can be used to approximate π (his approximation is around 3.142337). In astronomy, he proposed that the Earth moves around the Sun, that the stars are other suns orbited by other inhabited worlds, and that space is infinite. He published improvements on the predicted positions of the Sun, Moon, and planets based on Ptolemy's model of a geocentric universe. Nicolas' philosophical works focused on the theory of knowledge, in which he concluded that all human knowledge must be mere conjecture and wisdom only follows by understanding the extent of one's ignorance. His philosophical works include *De conjecturis* ("On Conjectures," 1440–44) and *Compendium* ("Collection," 1464). It does not seem that Nicholas' mathematical works are widely available today, but some of his philosophical work is available in English and several of his theological writings are available. For example, his philosophical work *De docta ignorantia* is in print in Jasper Hopkins' *Nicholas of Cusa on Learned Ignorance: A Translation and an Appraisal of De Docta Ignorantia*, 2nd edition (Arthur Banning Press, 1985). His works are in print in, among others, H. Lawrence Bond's translation *Nicholas of Cusa: Selected Spiritual Writings* (Paulist Press, 1997), and Emma Gurney Salter's translation *The Vision of God* (Cosimo Classics, 2007). The history in this note and the image below are from the MacTutor biography webpage on Nicholas of Cusa (accessed 6/24/2023).

Note 8.5.C. Georg ["of" or "von"] Peurbach (May 20, 1423–April 8, 1461) was an Austrian astronomer who contributed to trigonometry and wrote a text on practical computations. He received his bachelor's degree in 1446 and a master's degree in 1453 from the University of Vienna. There was no astronomer at the university while Peurbach was there, so he likely is self-taught in this area (he had access to the university library). He traveled between 1448 and 1453 lecturing on astronomy in Germany, France, and Italy. He was eventually appointed as court astrologer to the Holy Roman emperor Frederick III. At this point in history, astronomy and astrology are closely intertwined (this explains the repeated fascination with predictions of the locations of the planets, Sun, and Moon and the continued use of Ptolemy's predictive model; these two areas split sometime around 1600 with the work of Kepler and Galileo). Peurbach *was* a student of Nicholas of Cusa (of the previous note) and, in turn, *had* Regiomontanus (of the next note) as a student. Peurbach and Regiomontanus collaborated during the last eight years of Peurbach's life. In 1454 Peurbach completed *Theoricae Novae Planetarum* ("New Theories of the Planets") which covers Ptolemy's epicycle of planetary motion. He updated data tables for use with Ptolemy's theory in the prediction of planetary locations and the predictions of eclipses in his *Tabulae Ecclipsium* ("Tables of Eclipses," circa 1459). In 1456 he observed Halley's comet (not so-called at the time, because the namesake of the comet, Edmond Halley [November 8, 1656–January 14, 1742], was not yet born; in 1705 Halley predicted that the comet would return in 1758 and, when it did, it was named for him; Halley himself saw the comet in September 1682 but died before the 1758 return). Peurbach started working on a translation of Ptolemy's *Almagest* from Greek into Latin which would act as a shortened version of Ptolemy's work that would make a suitable teaching text, but he died before its completion. Regiomontanus completed it and it was published as *Epitome in Ptolemaei Almagestum* ("Summary of Ptolemy's Almagest," in 1462). His work on mathematics *per se* includes his *Tractatus super propositiones Ptolemaei de sinubus et chordis* ("A Treatise on the Propositions of Ptolemy concerning Sines and Chords"), an early work on trigonometry, and *Algorismus*, an algorithmic approach to computations with integers and fractions (this was popular and reprinted several times). The history in this note and the image below are from the MacTutor biography webpage on Georg Peurbach (accessed 6/24/2023).

Note 8.5.D. Johann Müller (June 6, 1436–July 6, 1476), better known as Regiomontanus, studied at the University of Leipzig from 1447 to 1450 (starting at the age of 11) and then at the University of Vienna from 1450 to 1452 where he had Peurbach as a teacher, and received a bachelor's degree. He joined the faculty of the University of Vienna in 1457 and started his collaboration with Peurbach. He worked on math, astronomy, he constructed astronomical instruments (such as

astrolabes) and taught classes, including one on Euclidean geometry. As mentioned in the previous note, Peurbach started work on a translation of Ptolemy's *Almagest*. However, he did not complete it and on his deathbed in 1461 he pleaded with Regiomontanus to complete the work (which he did and it appeared as *Epitome in Ptolemaei Almagestum* in 1462). Also in 1462, while in Venice, Regiomontanus found an incomplete copy of Diophantus' (circa 200–cica 284) *Arithmetica*. In fact, no complete version is known even today, but Regiomontanus brought Diophantus' *Arithmetica* to the attention of the European mathematical community. While working on his translation of *Almagest*, Regiomontanus realized the need for an organized approach to trigonometry in support of astronomy. He published *De triangulis omnimodis* in 1464 which gave methods for solving triangles. Structured along the line Euclid's *Elements*, the work consisted of five books, the first of which gives definitions, axioms, and 56 theorems on geometry. In Book II he starts the study of trigonometry and states his version of the Law of Sines and considers areas of triangles. In Books III, IV, and V he considers spherical trigonometry. A translation into English was made by Barnabas Hughes in *Regiomontanus On Triangles: De Triangulis Omnimodis* by Johannes Müller (University of Wisconsin Press, 1967). Some details on this work are given in *Sourcebook in the Mathematics of Medieval Europe and North Africa* edited by Victor Katz (Princeton University Press, 2016); see Chapter 1, "The Latin Mathematics of Medieval Europe" by Menso Folkerts and Barnabas Hughes, Section II, "A School Becomes a University: 1140–1480," Subsection II.5, "Trigonometry," Part 6, "Regiomontanus, *On Triangles* pages 162–173. In 1467 Regiomontanus computed a table of sine using sexagesimal numbers, and in 1468 computed a table of sines using decimals. He built an observatory in Nuremberg (Bavaria, Germany) and a workshop to construct instruments (this is before the invention of the telescope around 1600, so all instruments are based on making naked-eye measurements). In January 1472 he observed a comet that would be identified 250 years later as Halley's Comet. Regiomontanus set up his own printing press in Nuremberg is 1471–72 (following Gutenberg's invention of it in 1450) and became the first publisher of scientific literature on math, astronomy, and geography (including ancient and medieval works, and those from his own time). He published his own *Ephemerides* (which described how to determine longitude based on the position of the Moon, though the needed level of accuracy in determining the position of the moon did not exist at that time), which was used by both Christopher Columbus and Amerigo Vespucci. In 1475 Regiomontanus was summoned to Rome by Pope Sixtus IV (concerning calendar reform) and he died there on July 6, 1476. He likely died by plague, but "some accounts" indicate that he was poisoned. The history in this note and the image below are from the MacTutor biography webpage on Regiomontanus (accessed 6/25/2023).

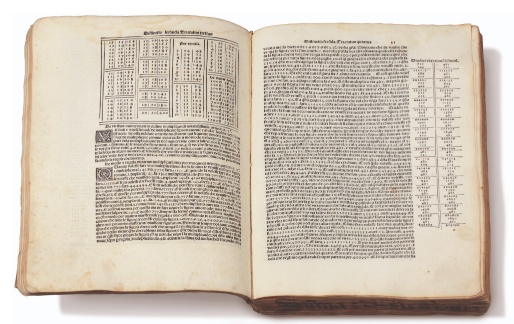
Note 8.5.E. Nicolas Chuquet (circa 1445–circa 1488) was the "most brilliant French mathematician of the fifteenth century" (according to Eves, page 266). He is known for one work, *Le Tripary en la science des nombres* ("The Science of Numbers in Three Parts"). In the closing sentences of this work, he states that he is a Parisian, has a Bachelor of Medicine, and that he wrote the book in Lyon (France), finishing it in 1484. Little more is known about his life, other than a few things revealed in Lyon tax registers of the time. *Le Tripary* is the earliest French algebra book, but it had little influence on the development of algebra and number systems and seems to have been lost. It was rediscovered by French linguist Eugène Aristide Marre (March 7, 1823–February 18, 1918) who published it in 1880. For some time it was thought that the first French algebra book was the 1520 *Larismetique noullellement composée* ("Arithmetic Newly Presented") by Estienne de La Roche (circa 1470–circa 1530), however it was found that much of de La Roche's book appropriates Chuquet's work (and not always with full understanding) with only two passages mentioning Chuquet. Though the printing press was available by 1484, *Le Tripary* remained in manuscript (i.e., hand copied) form. The first part, includes two chapters on arithmetic of whole numbers and fractions, and introduces the Hindu-Arabic numerals much in the way that Leonardo of Pisa does in *Liber abaci* (1202), but Chuquet also includes zero. The third chapter of the first part covers progressions and proportions. The fourth chapter considers arithmetical problems. The second part deals with the extraction of roots and irrational numbers. The third part deals with algebra, positive and negative integer exponents, and the solving of quadratic equations by completing the square. Chuquet deals with negative numbers as solutions to equations by interpreting them as debts. According to Eves (page 266) *Le Tripary* "was too advanced, for the time, to exert much influence on his contemporaries." The fact that it never went into print and that de La Roche's work duplicated it also must have had an impact on its influence. An English version of *Le Tripary en la science des nombres* is available as: *Nicolas Chuquet, Renaissance Mathematician: A Study with Extensive translation of Chuquet's Mathematical Manuscript Completed in 1484*, edited by Graham Flegg, Cynthia Hay, and Barbara Moss (D. Reidel Publishing Company, 1985). Some of the arithmetic problems from Chuquet's *Le Tripary* are given in Problem Study 8.9. The history in this note and the image below left are from the MacTutor biography webpage on Chuquet. The image of the book below right is from Amazon.com (both webpages accessed 6/25/2023).



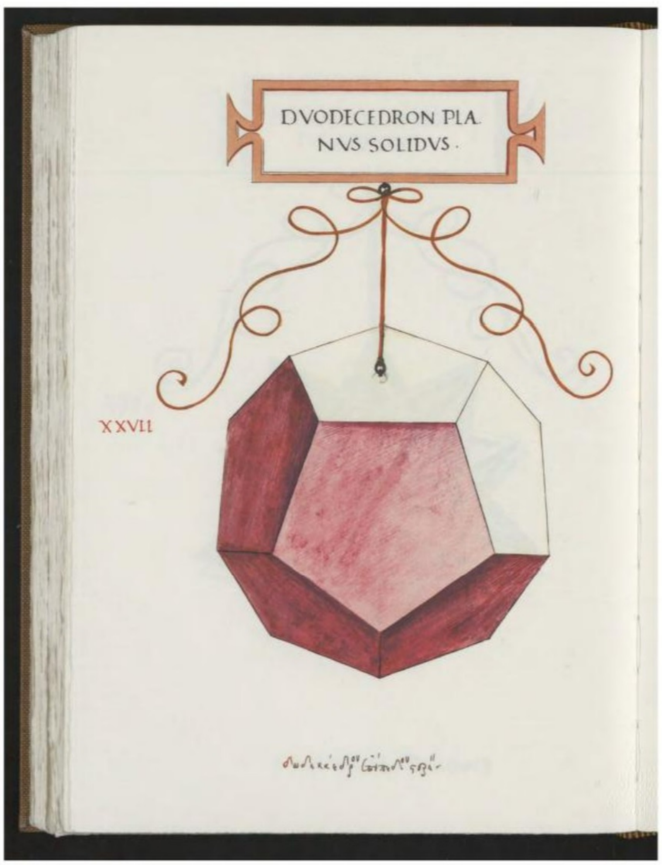
Note 8.5.F. Luca Pacioli (1445–1517), an Italian mathematician, wrote the most comprehensive math book of its time, *Summa de arithmetica, geometria, proportioni et proportionalita* ("Summary of Arithmetic, Geometry, Proportion, and Proportionality") in 1494. As we observed in Supplement. Leonardo of Pisa (Fibonacci) and the *Liber abbaci*, one line in *Summa* referencing "Leonardo Pisano" lead to the rediscovery of the significance of *Liber abbaci* and "Fibonacci" in the late 19th century by Pietro Cossali (June 29, 1748–December 1815). We also saw a figure concerning finger numbers from the *Summa* in Section 1.3. Finger Numbers and Written Numbers (the image is from page 13 of Eves). The *Summa*, at 600 pages, is encyclopedic in it coverage of arithmetic, algebra, tables of moneys, weights and measures, double-entry bookkeeping, and Euclidean geometry. It borrows freely from other sources and Eves claims (page 267): "It contains little of importance not found in Fibonacci's *Liber abaci*, but does employ a superior notation." However, it "was to provide a basis for the major progress in mathematics which took place in Europe shortly after [its] time" (quoting the MacTutor biography webpage on Pacioli). According to Alan Sangster, Gregory Stoner, and Patricia McCarthy in "The Market for Luca Pacioli's *Summa Arithmetica*," *Accounting Historians Journal*, **35**(1), 111–134 (2008) (this is available online on the Middlesex University (London) Research Repository; accessed 7/1/2023), "Even in the late 15th century, the notations used when writing mathematical computations and algebraic equations were not standardized and were far more cumbersome than today. . . . Pacioli introduced the symbols p̄ (for *piu* or "more," i.e. plus) and m̄ (for *meno* or "less," , i.e. minus) for the first time in a printed book, symbols that became standard notation in Italian Renaissance mathematics" (page 115). Eves further elaborates on the notation in *Summa* by observing that Pacioli uses *co* (for *cosa* or "thing") to represent an unknown variable such as x, *ce* (for *censo* or "square") for x, *cu* (for *cuba* or "cube") for x^3, *cece* (for *censo-censo*) for x^4, and *ae* (for *aequalis* or "equals") for = (see Eves' page 267). Sangster, Stoner, and McCarthy describe *Summa* and outline it contents as follows (on their page 121):

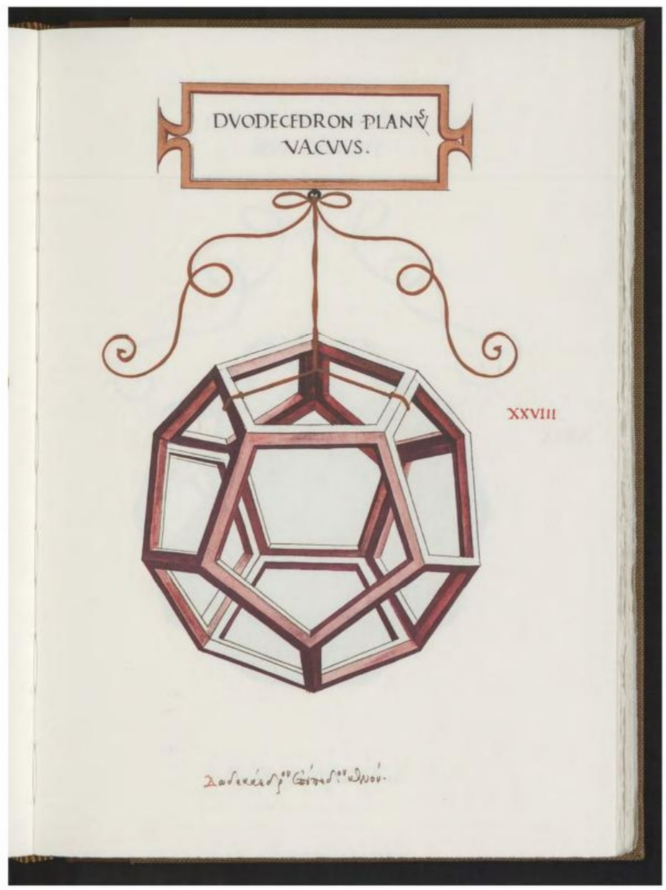
> "At 615 pages, [*Summa*] was a very large book, the equivalent of a 1,500 page textbook if typeset today. Its content clearly went beyond the level of the other *abbaco* texts [i.e., abacus books] , particularly the algebra. It was the complete technical manual for merchants and, for its day, contained the most comprehensive range of material available to meet their needs. After 16 introductory pages, the material in [*Summa*] is presented in ten primary chapters, printed and separately paginated into two volumes:
>
> *Volume 1*
>
> | | |
> |---|---|
> | 1 to 7 | on arithmetic (222 pages) |
> | 8 | on algebra (78 pages) |
> | 9 | on business (150 pages) divided into 12 sections, the first ten on various items relevant to business (including barter and bills of exchange), the eleventh on bookkeeping (27 pages), and the twelfth on weights and measures and exchange rates |
>
> *Volume 2*
>
> | | |
> |---|---|
> | 10 | on geometry and trigonometry (151 pages)" |

Eves states (on his page 167): "The algebra in the *Sūma* goes through quadratic equations and contains many problems that lead to such equations." The *Summa* is written in the vernacular (i.e., Italian, though the book title and section titles are in Latin), but surprisingly there is no English translation of most of the book (as Sangster, Stoner, and MacCarthy mention this in their 2008 paper on their page 111; a quick internet search reveals that this is still the case [as of July 2023]). However, the bookkeeping section, *Particularis de Computis et Scripturis* ("Particulars of Accounts and Writings"), introduces double-entry bookkeeping and this has generated sufficient interest that reprints of an English translation are still available. For example, John B. Geijsbeek's *Ancient Double-Entry Bookkeeping: Lucas Pacioli's Treatise* (A.D. 1494—the Earliest Known Writer on Bookkeeping) reproduced and translated with reproductions, notes and abstracts from Manzoni, Pietra, Mainardi, Ympyn, Stevin and Dafforne (originally published in 1914), Filiquarian Legacy Publishing (2012), is available in hardback, paperback, and Kindle; the original translation can also be viewed online on Archive.org (accessed 7/1/2023). A copy of *Summa* (in Italian, in its entirety) can be viewed on Google-Books (click on "Preview this book" to see the whole book; you can also download it by clicking on the gear icon; accessed 6/27/2023). As a bit of trivia, we observe that in 2019, a 1494 copy of *Summa* was sold for $1,215,000 (USD) by Christie's auction house (accessed 6/27/2023):



We now shift of biographical information about Pacioli. He was born in Sansepolcro in central Italy (near Florence). While still young, be moved to Venice to to tutor the sons of a wealthy merchant. While there, he continued his study of math and gained experience in business from his employer. He wrote his first work, a book on arithmetic, in 1470. Some years later he became a friar in the Franciscan Order. He taught at the University of Perugia from 1477 to 1489 where he wrote a second arithmetic book. He taught in Zara (or Zadar in modern-day Croatia) and wrote a third arithmetic book. Only the second of these books survives. In 1489 he returned home and worked on *Summa*. He traveled to Venice in 1494 and it was published there. That same year, Pacioli was invited to Milan to teach math in the court of the duke of Milan (Ludovico Sforza). While in Milan, Pacioli and Leonardo da Vinci (April 15, 1452–May 2, 1519) became friends and shared knowledge about math and art. This is when Pacioli started work on his second famous work, *Divina proportione* ("Divine Proportion'). He published the completed work of three volumes (or three "books") in 1509, with da Vinci as his illustrator! The first two books of the work focuses on the "golden ratio" or the *divine proportion*, which is the ratio $\varphi = a/b = b/(a+b)$ where $0 < b < a$. We have $\varphi = (1+\sqrt{5})/2$. For more on the golden ratio, see my online notes for the history component of Introduction to Modern Geometry (MATH 4157/5157) on Section 1.4. The Regular Pentagon; see, in particular, Note 1.4.A. These books also contain some theorems of Euclid on the ratio and on regular polygons, and applications of the divine proportion to architecture. The third book is a translation from Latin into Italian of Piero della Francesca's (June 1420–October 12, 1492; see also my online notes for Axiomatic and Transformational Geometry [MATH 5330] covering projective geometry on Section 1.1. Introduction and notice Note 1.1.E) *Libellus de quinque* *corporibus regularibus* ("Short book on the five regular solids"). There seems to be no translation of *Divina proportione* from Italian into English. There is a fictional novel by W.A.W. Parker titled *The Divine Proportions of Luca Pacioli: A Novel Based on the Life of Luca Pacioli* (Barbera Foundation, Inc., 2019), but this is not relevant. The 1509 Italian version is on Archive.org. The da Vinci illustrations of solid polyhedra are quite appealing and appear on pages 152 to 211. Two examples from Archive.org are given below.





In 1499, French armies entered Milan and Pacioli and da Vinci fled to Mantua (near Verona, Italy), then Venice, and finally Florence where they shared a house together. Pacioli taught geometry at the University of Pisa (which had been recently moved to Florence) from 1500 to 1506. Between 1501–02 Pacioli worked with Scipione del Ferro (February 6, 1465–November 5, 1526). Around 1515 del Ferro gave a solution to a cubic equation of the form $ax^3 + bx = c$ (we'll discuss this in more detail in Section 8.8. Cubic and Quartic Equations; you also may discuss it at the beginning of Introduction to Modern Algebra [MATH 4127/5127] in Supplement. A Student's Question: Why The Hell Am I In This Class?). Since Pacioli gave solutions of quadratic equations (and discussed cubic equations), it has been speculated that Pacioli and del Ferro may have discussed solving cubic equations. In 1509 (in addition to publishing *Divina proportione*) Pacioli also published a Latin translation of Euclid's *Elements*. As mentioned in Section 8.3. Fibonacci and the Thirteenth Century, Campanus of Novara (1220–1296) wrote a Latin translation of *Elements* around 1260. A printed version was published in Venice in 1482, and Pacioli's edition is based on this version, though Pacioli added many annotated comments to his version. Pacioli returned to Perugia in 1510 where he lectured, then he lectured in Rome in 1514. He returned to his birthplace of Sansepolcro and died there in 1517. He left unpublished a work *De Viribus Quantitatis* ("On the Power of Numbers") on recreational math problems and geometrical problems. The draft of the work referenced da Vinci and many of the problems from this draft appear latter in Leonardo's notebooks. Though Pacioli did not make original contributions to mathematics, he had an influence for a long period due to the widespread use of *Summa*. The history in this note and the image below left are from the MacTutor biography webpage on Pacioli. (accessed 7/1/2023).

Note 8.5.G. One final character from the 15th century we mention is German mathematician Johannes Widman (1462–1498). In 1489 he published (in German) the arithmetic book *Behende und hupsche Rechnung auf allen kauffmanschafft* ("Nimble and Smart Calculation for all Merchants"). The book appeared in print-form and is judged better than its predecessors due to its numerous useful examples. It had three parts, the first on counting with whole numbers, the second on proportion, and the third on geometry. In this book, the symbols "+" and "−" first appear. However, they do not represent the *operations* of addition and subtraction, but instead represent the *signs* of positive and negative (or "excess" and "defect"). A 1526 version of the book can be viewed on Google Books (accessed 7/2/2023; the following image is from this site).

This brief note is based on the MacTutor biography of Widman (accessed 7/2/2023) and John Mazur's *Enlightening Symbols: A Short History of Mathematical Notation and Its Hidden Powers* (Princeton University Press, 2014). See Mazur's page 162; this reference gives the history of mathematical symbols used from antiquity up to about 1800.

Revised: 10/24/2023