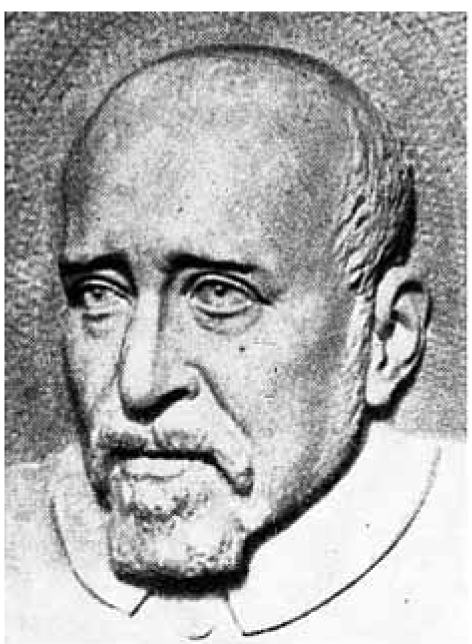
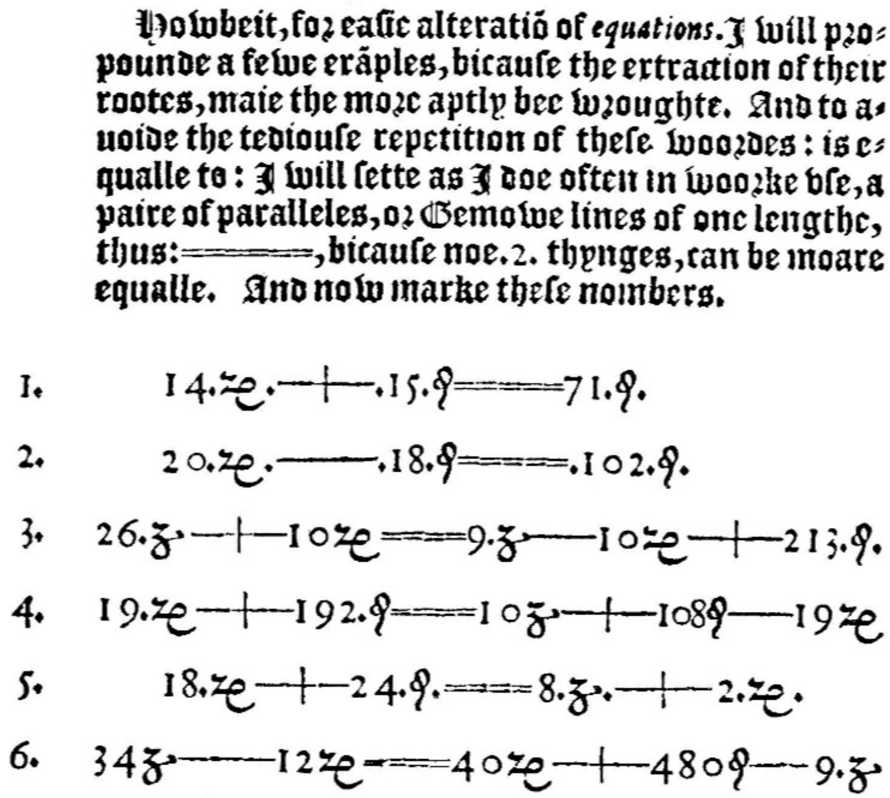
8.7. Beginnings of Algebraic Symbolism

Note. In this section we consider three works: *The Whetstone to Witte* by Robert Recorde (1557), *Die Coss* by Christoff Rudolff (1525), and the 1553 edition of *Arithmetica integra* by Michael Stifel. As observed in Section 8.6. The Early Arithmetics, the equals sign, "=," was introduced in *The Whetstone to Witte* (see Note 8.6.I). The *radix*, that is the square root symbol $\surd$, first appeared in *Die Coss*. Stifel's 1553 edition of *Arithmetica integra* used "M" and "D" for multiplication and division, respectively, represented the unknown by a letter, and often used the symbols $+$, $-$, and $\surd$.

Note 8.7.A. The last section, Section 8.6. The Early Arithmetics, concluded with a list of the mathematical works of Welsh doctor and mathematician, Robert Recorde (1510–1558). It is in his 1557 *The Whetstone of Witte* in which the equal sign, "=," first appears.



Part of page 238 of the Archive.org copy of *The Whetstone of Witte* is:



Howbeit, for easie alteratiõ of *equations*. I will propounde a fewe exãples, bicause the extraction of their rootes, maie the more aptly bee wroughte. And to auoide the tediouse repetition of these woordes : is equalle to : I will sette as I doe often in woorke use, a paire of paralleles, or Gemowe lines of one lengthe, thus:═════, bicause noe.2. thynges, can be moare equalle. And now marke these nombers.

1. 14.ze.—+—.15.φ═════71.φ.
2. 20.ze.———.18.φ═════.102.φ.
3. 26.ʒ.—+—10ze═════9.ʒ———10ze—+—213.φ.
4. 19.ze—+—192.φ═════10ʒ—+—108φ———19ze
5. 18.ze—+—24.φ.═════8.ʒ.—+—2.ze.
6. 34ʒ———12ze═════40ze—+—480φ———9.ʒ.

The last two sentences of text read (in modern English): "And to avoid the tedious repetition of the words 'is equal to,' I will let, as I do often in works, [this be denoted] by a pair of parallel lines of one length, thus: ═════, because no two things can be more equal. And now mark these numbers." As explained on page 146 of the Archive.org version, Recorde uses the following symbols:

φ this follows a number

ze this represents unknown x

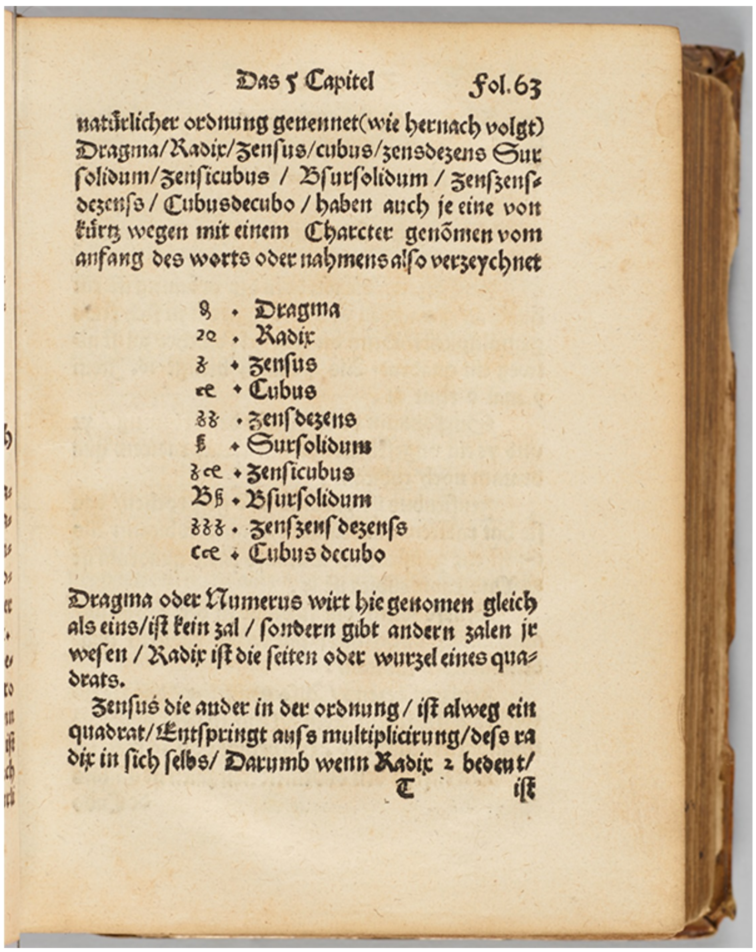
ʒ this represents unknown x^2

It is unclear as to the exact meaning of the periods. Therefore the equations given translate into modern notation as

1. $14x + 15 = 71$
2. $20x - 18 = 102$
3. $26x^2 + 10x = 9x^2 - 10x + 213$
4. $19x + 192 = 10x^2 + 108 - 19x$
5. $18x + 24 = 8x^2 + 2x$
6. $34x^2 - 12x = 40x + 480 - 9x^2$

An obvious stylistic difference from our time is the elongated nature of Recorde's addition, subtraction, and equal signs. This note and the image of Recorde above is from information on the MacTutor biography of Recorde (the webpages of this note were accessed 7/8/2023).

Note 8.7.B. Christoff Rudolff (1499–1543) was born in what today is Jawor, Poland (near Wrocław). His native language was likely German (since the city was under Bohemian control since 1335). He studied algebra in Vienna (Austria) sometime before 1521. Though he never worked for the university, he continued to use university facilities and gave private math lessons. He wrote *Behend vnnd Hubsch Rechnung durch die kunstreichen regeln Algebre so gemeinlicklich die Coss genent werden* ("A clever way of calculating sums by the artful rules of algebra commonly called 'the thing' "), which is usually referred to simply as *Die Coss*, in 1525 making it the first German algebra book (recall that Adam Riese's *Coss* was also written in 1525 in German, but remained in unpublished manuscript form). We have seen that "coss" was the term used in the middle ages for the unknown (see Note 8.6.G in Section 8.6. The Early Arithmetics). Rudolff observes that other early algebra texts use full words to represent an unknown, and he abbreviates this by using a single letter to represent an algebraic quantity. The book has two parts. The first part contains twelve chapters and covers arithmetic background necessary to understand the second part. The second part contains three chapters and covers the solving of linear and quadratic equations. Given in the first part are explanations of how to calculate with whole numbers and fractions, the Rule of Three, and methods to compute square roots and cube roots. He introduced the symbol ✓ for square roots (similar to the *radix* we use today, "$\sqrt{\ }$"; according to Eves on page 270, this symbol resembles a small r for *radix*). Similarly, he used ✓✓✓ to represent cube roots and ✓✓ to represent fourth roots (these second two symbols seem maybe out of order, but this is the notation as presented by Rudolph; these fonts are given in Joseph Mazur's *Enlightening Symbols: A Short History of Mathematical Notation and Its Hidden Powers* (Princeton University Press, 2014); see page 128. When dealing with square roots of more than one term, Rudolph introduces a period to indicate "take the square root of everything to the right." For example, ✓140 represents $\sqrt{140}$ and ✓.✓$140 + 12$ represents $\sqrt{\sqrt{140} + 12}$. When introducing algebraic expressions, Rudolff introduces new symbols for each of the powers 0 through 9 of the unknown. Whereas we would write $1, x, x^2, x^3, x^4, \ldots, x^9$, he uses the symbols which he has named dragma (or numerus), radix, zensus, cubus, zensdezens, ..., cubuc decubo, respectively. See the figure below. Rudolph considers adding, subtracting, multiplying, and dividing polynomial expressions, including sophisticated examples such as (in our notation): $\dfrac{12}{x+2} - \dfrac{x-2}{12} = \dfrac{148 - x^2}{12x + 24}$. In the second part, when considering quadratic equations earlier works had presented 24 different cases whereas Rudolph reduces this to 8 cases. Rules are given for solving equations, and the last chapter includes over 400 examples. *Die Coss* concludes by stating three cubic problems. Solutions are not given, as Rudolff states, because he wants to motivate additional algebraic research. *Die Coss* is still in print in German as *Die Coss Christoffs Rudolffs; Mit Schonen Exempeln Der Coss* (Rarebooksclub.com, 2012).


Das 5 Capitel Fol. 63

natürlicher ordnung genennet (wie hernach volgt) Dragma/Radix/Zenſus/cubus/zensdezens Surſolidum/Zenſicubus / Bſurſolidum / Zenſzenſdezenſs / Cubusdecubo / haben auch je eine von kürtz wegen mit einem Charcter genom̃en vom anfang des worts oder nahmens alſo verzeychnet

· Dragma
· Radix
ʒ · Zenſus
ce · Cubus
ʒʒ · Zenſdezens
ß · Surſolidum
ʒce · Zenſicubus
Bß · Bſurſolidum
ʒʒʒ · Zenſzenſdezenſs
cce · Cubus decubo

Dragma oder Numerus wirt hie genomen gleich als eins/iſt kein zal / ſondern gibt andern zalen jr weſen / Radix iſt die ſeiten oder wurzel eines quadrats.

Zenſus die ander in der ordnung / iſt alweg ein quadrat/Entſpringt auſs multiplicirung/deſs radix in ſich ſelbs/ Darumb wenn Radix 2 bedeut/

T iſt

Folio 63 of *Die Coss* from Frank Swetz's "Mathematical Treasures—Rudolff's Arithmetic and Algebra," *Convergence* (January 2021) (accessed 7/12/2023)

Rudolff finished *Künstliche Rechnung mit der Ziffer und mit den Zahlpfennigen* ("Calculating sums with the number and the number of pennies") in 1526. It consists of three parts. The first part introduces methods of computation with whole numbers and fractions, both with and without an abacus. The second part covers the Rule of Three and solutions of applied arithmetic problems. The third part gives around 300 problems related to commercial and manufacturing applications. In 1530 he published *Exempel Büchlin* ("Example Booklet"). The content is similar to the third part of his previous book. Its significance lies in the fact that it is the first work in which calculations are shown with decimal fractions. is important for it is the first work in which one finds calculations with decimal fractions (in a table of compound interest). His notation is conceptually identical to ours, but instead of a decimal point he used a vertical bar to separate the integral and fractional parts of a number. This note is based primarily on the MacTutor biographical webpage on Rudolff, with some reference to the MacTutor webpage on Earliest Uses of Symbols for Fractions (accessed 7/11/2023).

Note 8.7.C. Michael Stifel (or Styfel, Styffel, Stiefell, Stieffel, or Sifelius; 1487–April 19, 1567) "has been described as the greatest German algebraist of the sixteenth century" (Eves, page 270). He attended the University of Wittenberg (Germany, halfway between Berlin and Leipzig) where he earned a master's of art. In 1511 he was ordained in a monastery at Esslingen, but his liberal views and refusal to take indulgence money from the poor was problematic. He was attracted to Martin Luther's ideas (it was October 1517 that Martin Luther made public his 95 theses condemning the selling of indulgences, starting the Reformation). He came to fear for his life and fled the monastery in 1522. While on the run, he stayed with Martin Luter and the two became friends. After moving several times due to the turmoil of the Reformation, Stifel settled in Annaburg (not far from Wittenberg) in 1528. Life slowed down for him then, and he followed his interest in using numerology to deduce hidden meanings in religious documents. In 1532 he published the pamplet *Ein Rechenbüchlein vom Endchrist: Apocalypsis in Apocalypsim* ("An arithmetic book of the Antichrist: Revelation of the Apocalypse"; he concluded that the end of the world was near and that the pope was the Antichrist).

Between 1535 and 1547 he ran a parish in Holzdorf (very near Annaburg). While in Holzdorf, he was close enough to the University of Wittenberg to study math there. He studied Euclid, as well as works of Christoff Rudolff (1499–1543) and Adam Riese (1492–1559). During his 12 years in Holzdorf, he wrote three of his own math texts: *Arithmetica integra* ("Integral arithmetic"; 1544), *Deutsche arithmetica* ("German arithmetic"; 1545), and *Welsche Practick* ("Italian practice"; 1546). Religious wars broke out in 1546 and Stifel ended up fleeing in 1549. He then ran a parish near Königsberg (modern day Kaliningrad, Russia) starting in 1551 and began lecturing on mathematics and theology at the University of Königsberg.

In 1559 Stifel is a "University Master and priest" at the University of Jena (about 40 miles southwest of Leipzig, Germany). Stifel remained in Jena for the rest of his life and continued his exploration of numerology. Stifel is best known for his 1544 *Arithmetica integra* ("Integral arithmetic"), which he published in Latin. It consists of three books. The first book is on number theory, including triangular numbers and "magic squares." The second book covers irrational numbers as presented by Euclid in Book X of *Elements*. The third book covers algebra ("coss"). Stifel considers negative numbers, though he called them "absurd" and "fictitious" (foreshadowing the treatment of "imaginary" or *complex* numbers). He solves certain cubic and quartic equations (that is, polynomial equations of degree three and four, respectively). For the first time, he (almost) considered exponents. While dealing with a geometric sequence, such as $2, 4, 8, 16, 32, \ldots$, he proposed extending it "backward" by adding $1, 1/2, 1/4, 1/8, \ldots$. Throughout, he uses the standard symbols $+$, $-$, $\surd$, and uses letters to represent unknowns. You can see a copy of *Arithmetica integra* in its original Latin on the Hathi Trust Digital Library and on Google Books (accessed 7/13/2023). *Deutsche arithmetica* ("German arithmetic"; 1545) was written, not surprisingly, in German. Its target audience, then, was not restricted to academics but was meant for wide range of readers. He introduced separate symbols for the powers of the unknown (similar to Rudolff's approach in his *Die Coss*, as described in Note 8.7.B). In 1553 Stifel published a new edition of Christoff Rudolff's *Die Coss* (see Note 8.7.B). Stifel's edition was over twice as long as Rudolff's, and Stifel added his own material. In particular, he considered powers of $(1 + x)$ and Pascal's Triangle (100 years before Pascal). In this work, he came closer still to an exponential notation. For unknown A, he represented its powers as $A, AA, AAA, AAAA, \ldots$. He used $\surd$ for square roots, but he did not have the full notation for nth roots $\surd$, $\sqrt[3]{}$, $\sqrt[4]{}$, etc., though he did take first steps in this direction. This note is based on the MacTutor biography webpage on Stifel (accessed 7/12/2023).

Revised: 7/13/2023