Supplement. Additional Numeral Systems

Note. In this supplement we explore several numeral systems which are not covered in Chapter 1, Numerical Systems, of Howard Eves' *An Introduction to the History of Mathematics* 6th edition (Saunders, 1990).

Note. We start with numeral systems of the Aboriginal Australians. As we commented in Section 1.2. Number Bases, a common myth is that the indigenous Australians counted as "one, two, three, many" and that they did not count past four or five. In *The Universal History of Numbers* (John Wiley & Sons, 2000), Georges Ifrah gives as a reference for this myth Alf Sommerfelt's *La Langue et la Societé* ["Language and Society"] (Oslo, 1938); see Ifrah's page 5. "Indigenous knowledge keeper" Shannon Foster addresses this myth in an online opinion piece on the University of Sydney wesite, Explainer: how does the Aboriginal numeric system work? (written February 1, 2017; accessed 6/5/2023). She says that there are many myths concerning the knowledge of the Indigenous Australians which range "from the ridiculous to the downright racist," including the myth that they don't count beyond four or five. She explains that it was often linguists who addressed Aboriginal numeracy and that they struggled "to understand Aboriginal people's expression of numbers." The Australian anthropologist Alfred Howitt (April 17, 1830–March 7, 1908) studied the Wurunjjeri people of Victoria (in southeastern Australia) and found that after counting to five on their fingers, they continued counting up the arm. They were then able to count to relatively high numbers *non-vocally* using body-counting (or "body-tallying," as Foster calls it); this help explain how linguists could fail to understand the counting technique. Foster goes on to consider how words were introduced representing body-count numbers and states that many of these (new) words for numbers above five are based on either corresponding English words or on the shapes of the Hindu-Arabic numerals.

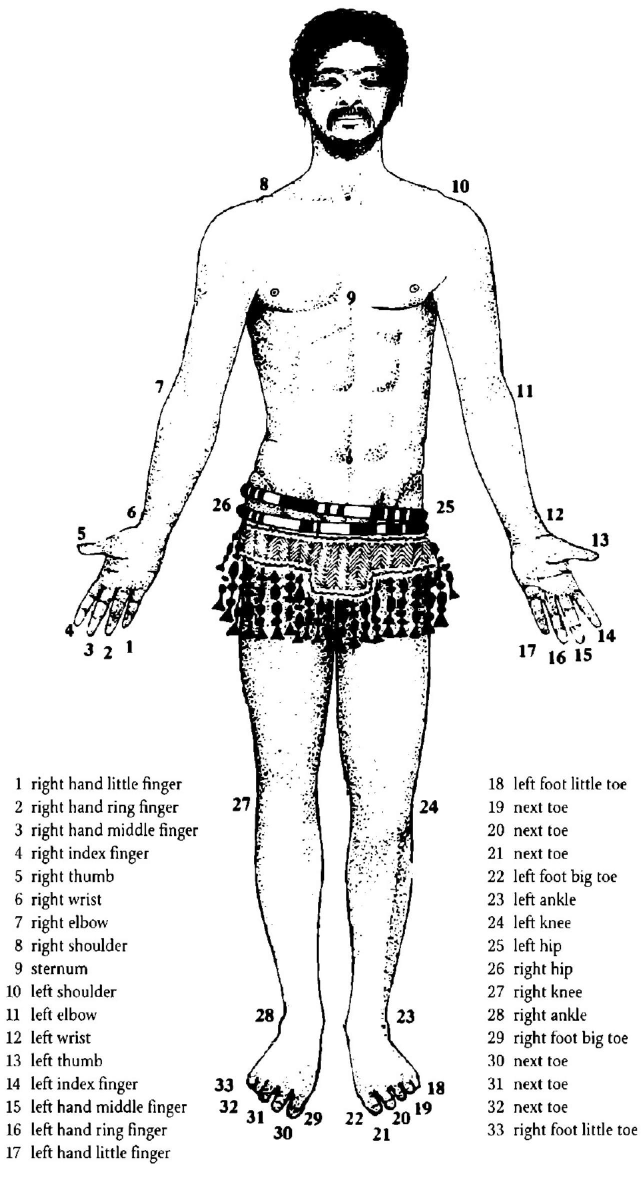


FIG. 1.30. *Body-counting system used by Torres Straits islanders*

In Section 1.2. Number Bases, we mentioned the Torres Strait Islanders who make up part of the indigenous Australian population (as an example of a culture with a small number base; see Note 1.2.A). Ifrah (on his page 12) has Wyatt Gill (an English missionary in Australia and the South Pacific; December 27, 1828–November 11, 1896) saying that this population "counted visually." This is illustrated in Figure 1.30 of Ifran, given above. Someone would touch a part of the body and it would indicate the number given in Ifran's Figure 1.30. Ifran also gives several other such diagrams, in particular for body-counting schemes used in New Guinea.

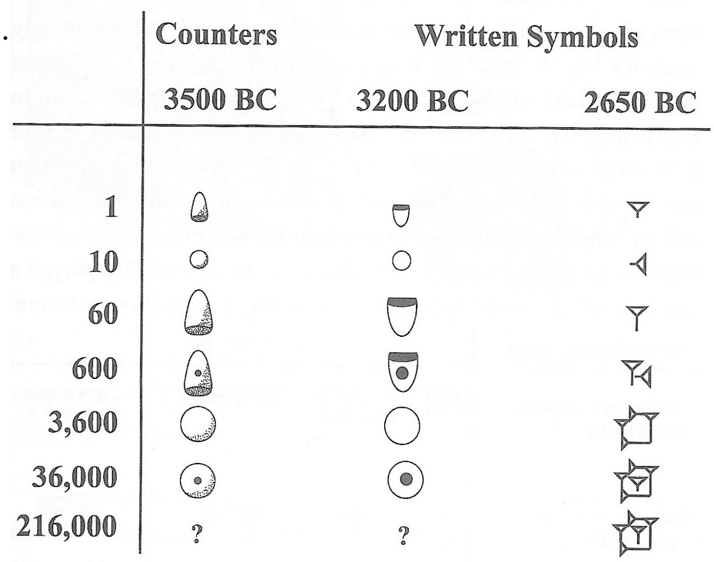
Note. Mesopotamia is the region of western Asia ranging from the Persian Gulf, through modern-day Iraq, and including parts of northeastern Syria. It is dominated by the Fertile Crescent between the Tigris and Euphrates rivers (though the Fertile Crescent also includes Syria, Lebanon, Israel, Palestine, and Jordan, giving it the crescent shape). Mesopotamia is sometimes called "the cradle of civilization," and is home to some of the earliest known examples of cursive script, mathematics, and agriculture.

Image from "The Latin Library" website on Sumer (accessed 6/6/2023)

The earliest writing from this area dates between 3500 BCE and 3000 BCE. The region was partitioned into several independent city-states (with frequent conflicts between them). The region was dominated by the Sumerians from sometime in the 4th millennium BCE to the 18th century BCE, when the Babylonians took control of the region with the city of Babylon (in southern Mesopotamia) as their political center. The "Old Babylonian Empire" (often abbreviated "OB") ranged from the 18th century BCE to around the 7th century BCE when the Neo-Babylonian Empire took control and reigned until 539 BCE when it was conquered by the Persian Empire. We considered the Babylonian cuneiform numerals in Section 1.4. Simple Grouping Systems, so we concentrate on the Sumerian approach now. This note is based on Peter Rudman's *How Mathematics Happenened: The First 50,000 years* (Prometheus Books, 2007) and Chapter 8, "Numbers of Sumer," in Georges Ifrah's *The Universal History of Numbers* (John Wiley & Sons, 2000). As opposed the the older pebble-counting techniques of antiquity, the Sumerians approached counting differently. (By the way, the Romans still used pebbles, or "*calculi*," and this is where our words "calculation" and "calculus" come from; Rudman, page 70.) The archaeological evidence from before 2000 BCE shows the use of clay counters of different sizes and shapes (a small clay cone to represent one, a clay ball for ten, a large cone for sixty, ...) replacing simple pebbles. Clay counters have been found dating as far back as the ninth millennium BCE, and in Sumer they have been found dating from around 3500 BCE (Rudman page 91). An excellent website on Sumerian counters (or "tokens") and clay tablets is Denise Schmandt-Besserat's Tokens: their Significance for the Origin of Counting and Writing (accessed 6/6/2023). On this website, it is stated that tokens were often perforated to be strung together (you may see counters with holes on them online), and others were kept in "envelopes" (which were hollow clay spheres). Some of the envelopes had symbolic impressions of the counters they held. This is illustrated in Shmandt-Besserat's Figure 3 below, where examples of counters are given, and a large engraved envelope is seen. These items are from Susa, Iran (in eastern Mesopotamia) from 3300 BCE.

Figure 3 from Denise Schmandt-Besserat's website on tokens

When the Sumerians started writing around 3000 BCE, the written symbols mimicked the shapes of clay counters (see the table below from Rudman's *How Mathematics Happened*).



	Counters	Written Symbols	
	3500 BC	3200 BC	2650 BC
1			
10			
60			
600			
3,600			
36,000			
216,000	?	?	

Table 3.2.3 The evolution of number symbols in Sumer

The Sumerian numerals were additive. They later evolved into a cuneiform style, which the Baylonians later modified into their positional numeral system (as discussed in Section 1.7. Positional Numeral Systems). These are compared in the next table (also from Rudman's *How Mathematics Happened*).

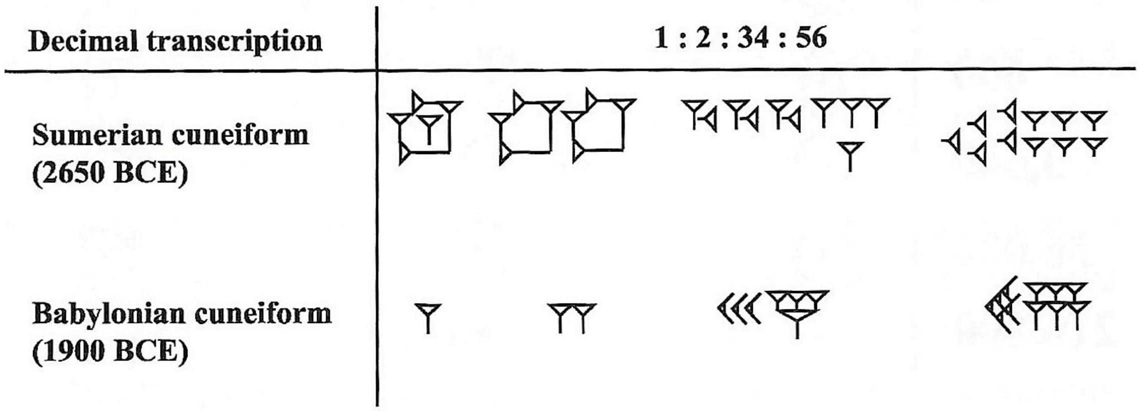


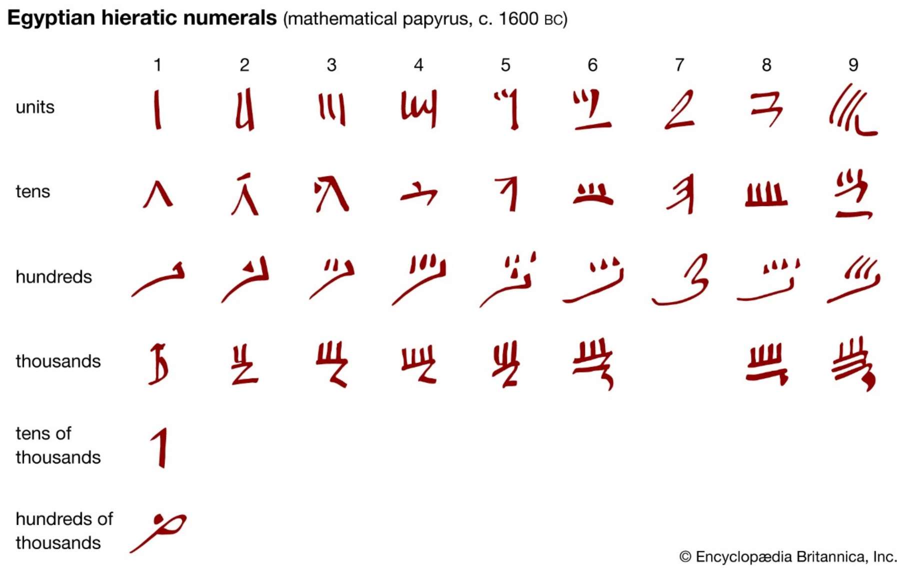
Table 3.2.4 *The transition of numbers from Sumerian additive to Babylonian positional*

Some comments on the development of Sumerian writing is appropriate here.

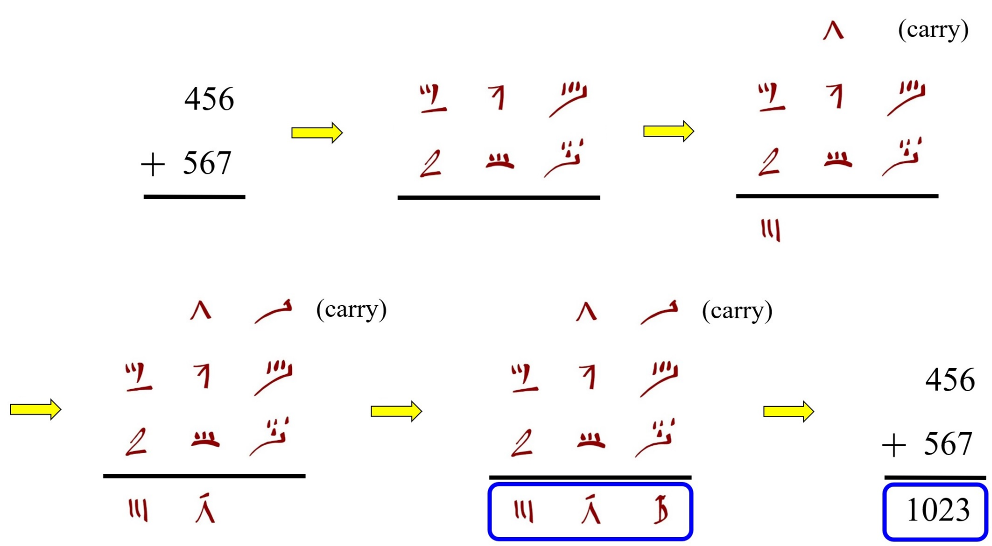
> "Writing, as a system enabling articulated speech to be recorded, is beyond all doubt among the most potent intellectual tools of modern man. Writing perfectly meets the need (which every person in any advanced social group feels) for visual representation and the preservation of thought (which of its nature would otherwise evanesce [be quickly forgotten]). . . . Writing, therefore, in revolutionising human life, is one of the greatest of all inventions. The earliest known writing appeared around 3000 BCE, not far from the Persian Gulf, in the land of Sumer, which lay in Lower Mesopotamia between the Tigris and Euphrates rivers." (Ifrah, page 77)

Like the Babylonians after them, the Sumerians "wrote" by making marks on soft clay tablets that were later hardened. Stone is rare in the Mesopotamian region, and wood, leather, and parchment does not preserve well. The soil of the region consists of clay alluvial deposits (i.e., deposited by water), so it is in great abundance and became an obvious medium for writing (as well as sculpting and construction). The numerical markings were often accompanied by an outline drawing (or pictograph) of a particular item, seeming to indicate that these are records of supplies, inventories, or exchanges (Ifrah, page 78). The number of estimated symbols used in Mesopotamia at this stage (circa 3000 BCE) was about 2000. However, this falls short of a true written language, since it depended on drawings of objects and could not recreate the whole spoken language. A transition from pictographs to symbols with phonetic meanings began around 2800 to 2700 BCE. The use of picture-signs to reflect different sounds made in Sumerian language meant that any ideas that could be expressed verbally, could be expressed using the written symbols. The Sumerian system was no longer limited to simply representing a few objects for which there are pictographs (hence using the written tablets to assist the memory, such as recalling how many containers of grain is in a shipment), but could be used to communicate complete thoughts and hence used to inform and instruct (Ifrah, pages 80 and 81)!

Note. We introduced the Egyptian hieroglyphic numeral system (from roughly 3000 BCE to 1000 BCE in Section 1.4. Simple Grouping Systems. Hieroglyphic writing was mostly used for inscriptions on stone monuments, tombs, and temples. Papyrus is similar to thick paper, papyrus is made from the pith of the papyrus plant, which was abundant along the Nile River delta; of course it does not survive the centuries as well as stone or clay. A cursive script, called *hieratic*, arose to replace hieroglyphics when writing on papyrus. It was used from around 2900 BCE to 200 BCE. One version is given in the figure below from the Britannica.com page on Mathematics in Ancient Egypt.



Even though this is derived from the Egyptian hieroglyphic numeral system, it involves more symbols and hence is more efficient at representing numbers. Hieratic numbers are written from right to left, as are the other Semitic languages (a collection of languages in the "Afro-Asiatic" language family, including Hebrew, Aramaic, and Arabic). Rudman's "Fun Question 3.1.1" in *How Mathematics Happened* is to convert 456 and 567 into hieratic symbols and then add. Here's the solution:



Note. We saw the alphabetic Greek numeral system in Section 1.6. Ciphered Numeral Systems. We now consider a similar system, the alphabetic Hebrew numerals. This note is based on the Wikipedia page on Hebrew Numerals (accessed 6/6/2023). The system dates from the late second century BCE, is very similar to the alphabetic Greek numeral system, and also has no symbol for zero. There are symbols for the units 1 through 9, the multiples of 10, 10 through 90, and the first four multiples of 100, 100 through 400. For the other multiples of 100, namely 500 through 900, the letter representing the some of the two or three letters representing the first multiples of 100 which add to give the result are used (though the Wikipedia page seems to incorrectly display this).

1	א	alef	10	י	yod	100	ק	qof
2	ב	bet	20	ך	kaf	200	ר	resh
3	ג	gimel	30	ל	lamed	300	ש	shin
4	ד	dalet	40	ם	mem	400	ת	tav
5	ה	he	50	ן	nun	500	תק	
6	ו	vav	60	ס	samekh	600	תר	
7	ז	zayin	70	ע	‘ayin	700	תש	
8	ח	chet	80	ף	pe	800	תת	
9	ט	tet	90	ץ	tsadi	900	תתק	

The Hebrew fonts here are produced with the LaTeX package `cjhebrew` (see John D. Cook's website for details; accessed 6/6/2023).

Note. Sequoyah (circa 1770–August 1843), also known as George Gist or George Guess, was born in Tuskegee, Cherokee Nation. The location of Tuskegee is in modern-day Vonore, Tennessee in Monroe County (the Birthplace of Sequoyah is located in Vonore, adjacent to the Fort Loudon State Park) and is about 35 miles southwest of Knoxville.

From the Britannica.com webpage on Sequoyah (accessed 6/6/2023)

Sequoyah created a "syllabary"(a set of symbols representing syllables of a language) in the late 1810s and early 1820s for the Cherokee language. Surprisingly, he was illiterate before starting this project! To do so, he introduced 85 characters. This made the Cherokee one of the first North American Indigenous groups with a written language. The numerals that Sequoyah introduced are given below. He gives a unique numeral for each of 1 through 20, and then gives numerals for the multiples of 10 of 30 through 90. As we'll see below, this hybrid system makes addition and "carrying" a bit tricky.

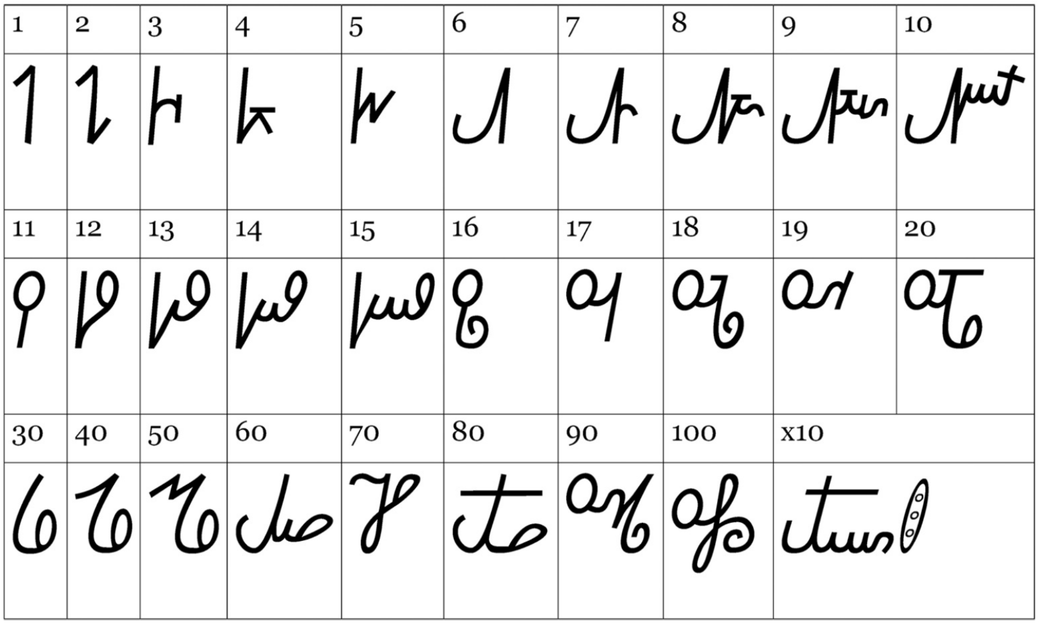
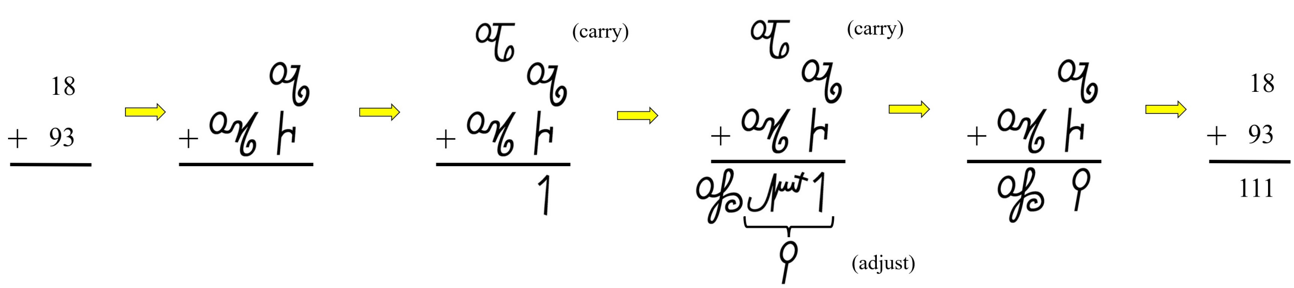
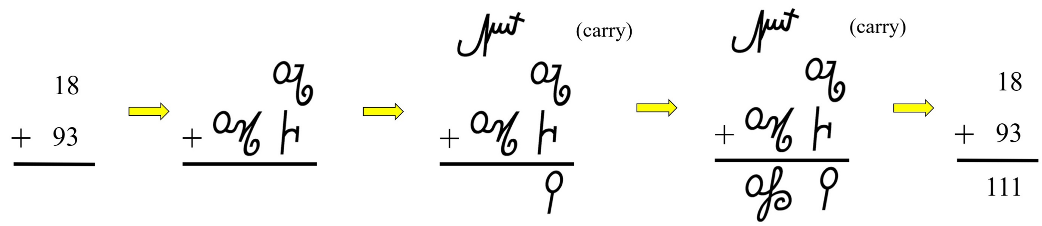


Figure 5.1 from Stephen Chrisomalis' *Reckonings: Numerals, Cognition, and History* (MIT Press, 2020)

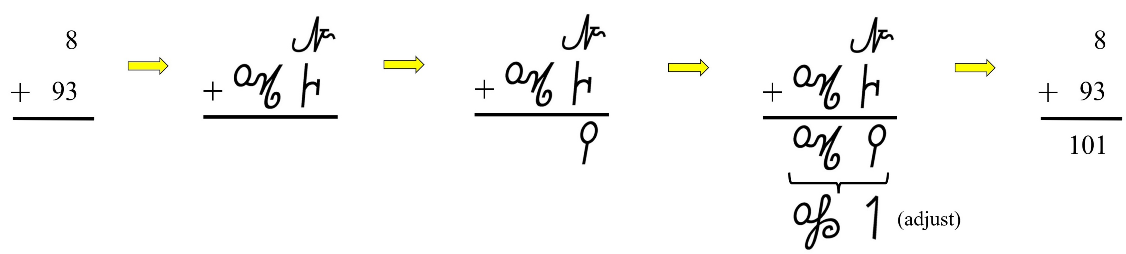
This historical information is from the Wikipedia webpages on Sequoyah and the Cherokee Syllabary (accessed 6/6/2023). As a quick example, we add 18 and 93 to illustrate some of the trickiness of their use:



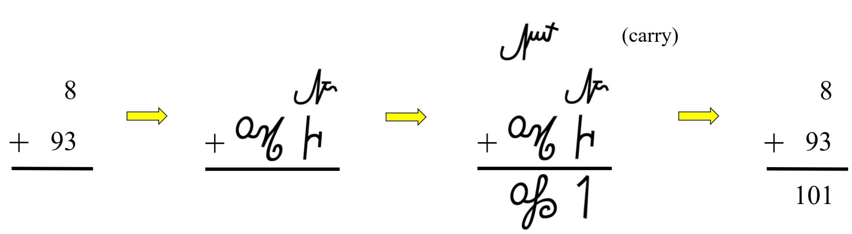
Notice that adding 18 and 3 yields 21, and if we bring down the 1 and carry the 20, the sum in the second column results in 110. The 110 is properly represented, but since the right-most two numerals add to give 11, they should be replaced with the single numeral representing 11, requiring the given adjustment. With this knowledge, we see that we might want to think ahead so that for the 21, we bring down 11 and carry 10. This gives the following:



On the other hand, if we consider $8 + 93$ and bring down the 11 (so that there is then nothing to carry), we get the two numeral representation of the sum 101 as "90 and 11." However, the numerals for 10 through 19 are used only on their own and not in addition with other numerals which are multiples of 10 and between 20 and 90. In other words, the "90 and 11" needs to be adjusted to "100 and 1."



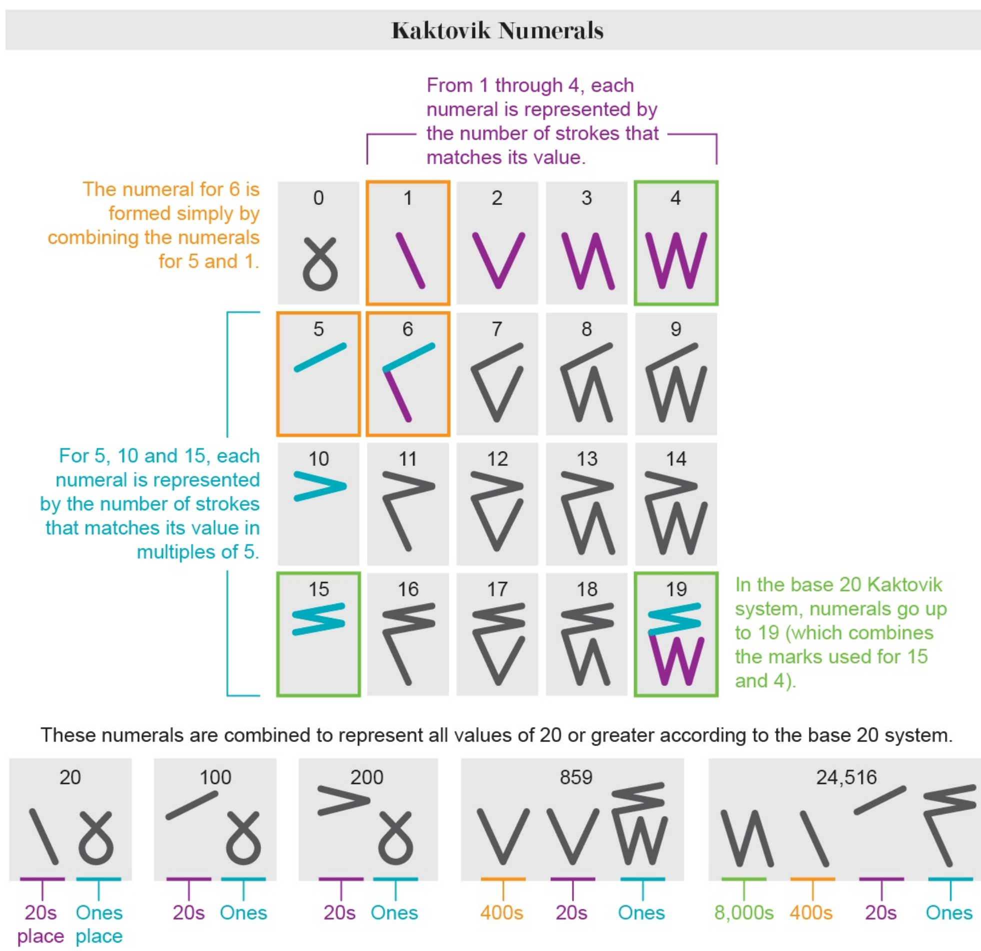
So we see that using the largest numeral that gives us the proper ones-value is not always the correct approach. In this case, we add the 8 and 3 to get 11, bring down the 1 and carry the 10, as follows:



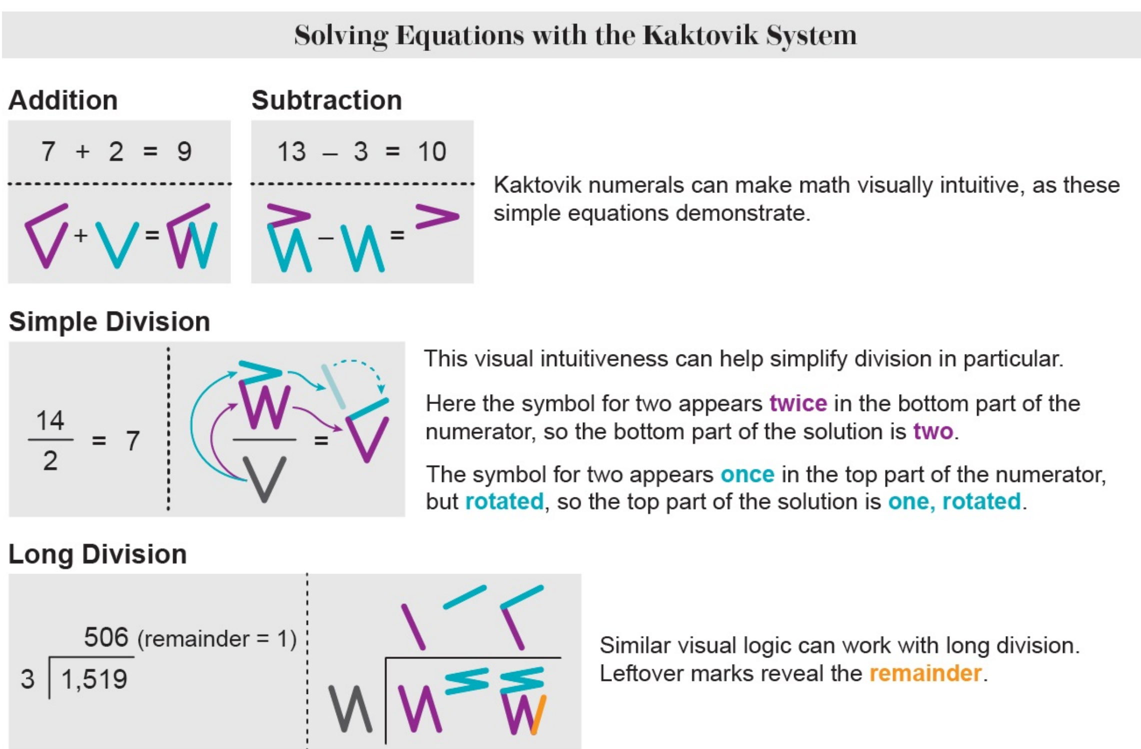
The choice of the numeral to bring down and the one to carry is affected by the numerals in the second column. On the next page is a poster from the Cherokee Nation Language Department webpage which gives the Cherokee numbers 1 to 100, along with a "new" numeral for zero.


ᏣᎳᎩ ᏗᏎᏍᏗ
Cherokee Nation
Language Technology

Note. A very recent numeral system (from about 30 years ago; we are referencing a 2023 article from *Scientific American*, on which this note is heavily based) was created by Inuit middle school students in the village on the north slope of Alaska of Kaktovik ("Qaaktuġvik" in Iñupiaq; the population of Kaktovik was reported as 283 in the 2020 census). This is the first new numeral system invented in the Western Hemisphere in over a century. Boosted by Silicon Valley, the numerals are planned to be available on smartphones and computers. These notes and the image below are based on the *Scientific American* website "A Number System Invented by Inuit Schoolchildren Will Make Its Silicon Valley Debut" (accessed 6/5/2023). The numerals (from this website) are as given in the next figure.



According to Nuluqutaaq Maggie Pollock, an instructor of the Kaktovik numerals in the town of Utqiaġvik (formerly "Barrow, AK"; population 4900 in the 2020 census and most populace city on the north slope of Alaska; near Point Barrow, the USA's northernmost land), the numeral system "is really the count of your hands and the count of your toes." That is, it is similar to the finger and toe body-counting which motivates a vigesimal scale, as mentioned in Section 1.2. Number Bases. For example, *Iñuiññaq*, the word for 20, represents a whole person, as Pollock explains. American schools notoriously suppressed native American culture, dress, and language during the 19th and early 20th centuries. The Inuit and the Iñnupiaq language were no exception. The Inuit numerals started as a class project and were meant to look like the Iñupiaq word they represent. There was no Iñupiaq word for zero, so the class invented a symbol for zero and the resulting symbols gave a base 20 numeral system.



The students were inspired by tally marks in the creation of the numerals, and this results in a geometric approach to addition, subtraction, and division (unlike the Hindu-Arabic numerals). Examples given on the *Scientific American* website are in the figure above. William Bartley, the teacher who helped develop the numerals, reported that, after a year of the students working with both the Kaktovik numerals and the Hindu-Arabic numerals, scores on standardized math exams jumped from below the 20th percentile to significantly above average. However, the Kaktovik numerals were pushed into a marginal role because of the federal No Child Left Behind Act (in force from 2002 to 2015). It was pushed out of the math classroom and into the Iñupiaq language classroom. This illustrates that the near universal use of the Hindu-Arabic numerals has resulted in the suppression of other culturally significant systems. However, with the numerals getting the support of Silicon Valley and "going digital," they have the potential for staying-power! As Nuluqutaaq Pollock puts it: "This is just the beginning."

Revised: 1/22/2024