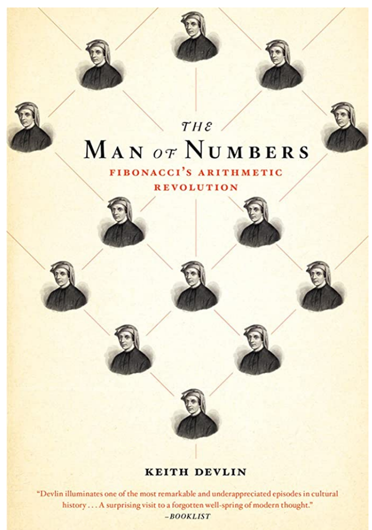
Supplement. Leonardo of Pisa (Fibonacci) and the *Liber abbaci*

Note. In this supplement, we present the history of Leonardo of Pisa (also known as Fibonacci) and his work, concentrating on his *Liber abbaci*. This supplement is largely based on Keith Devlin's *The Man of Numbers: Fibonacci's Arithmetic Revolution*, Walker and Company (2011). You might even think of this supplement as a bit of a "book report" on Devlin's work.



Note. Some common myths about Leonardo of Pisa include: (1) he was known as Fibonacci, (2) his book *Liber Abbaci* (we follow this spelling used by Devlin in these notes, though this is not universal) lead to the quick spread of the Hindu-Arabic numerals throughout Europe, and (3) he was quickly recognized for his place in the history of mathematics. The name "Fibonacci" was attached to Leonardo by historian Guillaume Libri in 1838. As we saw in Section 1.9. The Hindu-Arabic Numeral System, it was centuries after *Liber abbaci* appeared before the Hindu-

Arabic numerals were widely accepted in Europe. As we observed in Section 8.3. Fibonacci and the Thirteenth Century, *Liber abbaci* did not appear in printing press version until 1857 when Baron Baldassarre Boncompagni printed a Latin version with no commentary. A printed English version (of length 672 pages) by Laurence Sigler was published by Springer in 2002 and is the only translation of *Liber abbaci* into a modern language!

Image from the MacTutor History of Mathematics Archive biography of Fibonacci (accessed 5/1/2023)

Above is the image of Leonardo of Pisa commonly presented. It is by an unknown artist and appeared in *I Benefattori dell'Umanità* ("The Benefactors of Humanity"), Vol. VI; Ducci, Florença, 1850; this reference is from the Leonardo Pisano, A.K.A. Fibonacci webpage of Fibonacci.com. Given the fact that it was drawn 600 years after he lived, it is unlikely to resemble Leonardo.

Note. Very little is known about the life of Leonardo of Pisa. From a legal document, it is known that his father's name was Guilichmus, which translates as 'William' (the variant Guilielmo is also common), and that he had a brother named Bonaccinghus (Devlin, page 9). In the second paragraph of Sigler's translation of *Liver abbaci* (Sigler, page 15) in the Prologue, we have the following brief biography as written by Leonardo himself:

> "As my father was a public official away from our homeland in the Bugia [now Bejaïa, in Algeria] customshouse established for the Pisan merchants who frequently gathered there, he had me in my youth brought to him, looking to find for me a useful and comfortable future; there he wanted me to be in the study of mathematics and to be taught for some days. There from a marvelous instruction in the art of *the nine Indian figures* [emphasis added], the introduction and knowledge of the art pleased me so much above all else, and I learnt from them, whoever was learned in it, from nearby Egypt, Syria, Greece, Sicily and Provence, and their various methods, to which locations of business I travelled considerably afterwards for much study, and I learnt from the assembled disputations."

This is the extent of the historical information on Leonardo. Consensus from several sources seems to be that he was born in 1170 and died in 1250. Devlin states (page 27): "We know he was born sometime around 1170 CE, but we do not know the exact year, and we are not completely sure where. Most likely, it was in Pisa, but in any event that was where he spent most of his childhood." Pisa (in present-day Italy) is home of the "Leaning Tower of Pisa," which was under construction (on and off) between 1173 and 1372. So Leonardo would have been exposed to the construction during his time childhood.

Note. Western history is often divided into three divisions: classical antiquity (from the 8th century BCE to the 5th century CE when the Roman Empire fell; this era is dominated by the ancient Greek and Roman civilizations), the medieval period (or "middle ages" from the 5th century to the late 15th century), and the modern period from 1500 CE to present). These divisions are further subdivided. The medieval period is broken into the early medieval period (or "early middle ages," from 1000 to 1300), the high medieval period, and the late medieval period (from 1300 to 1500). The early medieval period is sometimes called "The Dark Edges" (not so much a term used by academics today). This period saw a population decline, a decline in trade, and an increase in migration over the past. There was a relative scarcity of literature and cultural output. Now this is in Europe; recall that we have seen intellectual growth (including in math) elsewhere in the world (namely, the Arab world and India; see Section 1.9. The Hindu-Arabic Numeral System). The high medieval period saw population and economic growth and the first universities. It also saw numerous wars in Europe and plagues. The "Black Death" resulted in the death of an estimated 75 to 200 million people in Europe, West Asia, and North Africa (much of this happened in the 1300s in the early part of the high medieval period). This history is based on the Wikipedia pages on the Early Middle Ages, the High Middle Ages, the Late Middle Ages, and the Black Death. So Leonardo lived his life as European society began to develop and prosper once again. Also at this time (and in Pisa and nearby cities) scholars were translating classical Greek works of Euclid, Apollonius, Archimedes, Aristotle, and Ptolemy into Latin (Devlin, page 30).

Note. Economic growth was in full swing during Leonardo's time, with banks giving loans and issuing letters of credit. This required numbers. Merchants had to measure their items, negotiate prices, and taxes had to be calculated on imports. To perform such computations, either fingers or an abacus (i.e., a counting table) were used. Then a written record of the computation was made, with no record of *how* the computation was done; they did not show their work. To check the work, the computation had to be repeated in its entirety. Leonardo clearly had these applications in mind when he wrote *Liber abbaci* (first published in 1202). He explained the concepts behind commercial applications and presented numerous examples to illustrate the ideas (Devlin, page 35).

Note. The relevant mathematical work which we know of today that predates Leonardo includes (Devlin, pages 47 and 48):

1. *Kitab al-fusul fi'l-hisab al-hindi* (Book of chapters on Hindu arithmetic) of Abu'l-Hasan Ahmad ibn Ibrahim al-Uqlidisi, written in Damascus in 952-53 CE. This is Arabic work on Hindu arithmetic (the oldest such surviving), but the only surviving version is a copy written in 1186 (two centuries later). This work is described in A. S Saidan, "The Earliest Extant Arabic Arithmetic: *Kitāb al-Fusūl fī al Hisāb al-Hindī* of Abū al-Ḥasan, Ahmad ibn Ibrāhīm al-Uqlīdisī, *Isis*, **57**(4), 475–490 (Winter, 1966).

2. *Liber ysagogarum alchorismi* (The Book of the Introductions of al-Khwārizmī") which may have been written by Adelard of Bath (1075–1160). It consists of sections on arithmetic, music, geometry, and astronomy. The first three books,

on arithmetic, cover the Hindu-Arabic arithmetic. These have been translated into French by André Allard, with commentary, in *Le calcul indien (Algorismus): histoire des textes, édtion critique, traduction et commentaire des plus anciennes versions latines remaniées du XIIe siècle*, Paris and Namur (1992). A translation into English of the geometrical part is given in Charles Burnett, "The Geometry of the *Liber Ysagogarum Alchorismi*," *Sudhoffs Archiv*, **97**(2), 143–173 (2013) (where the reference on Allard's work is given).

3. *Arithmetic* by al-Khwārizmī (790–850). The oldest surviving copy of this is a Latin manuscript. It covers place-value use of numerals. It seems likely that Leonardo could read Latin, since this would be necessary to access sources such as this.

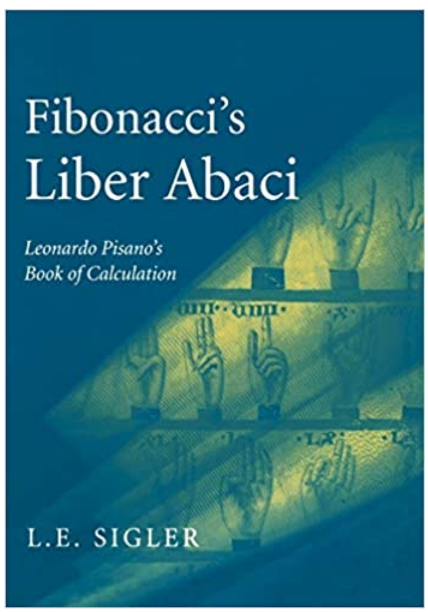
4. *Algebra* by al-Khwārizmī. Leonardo based his treatment of algebra in *Liber abbaci* on this. He relied heavily on this source, almost certainly from a Latin translation (Devlin, page 48). It was translated by, among others, Gerard of Cremona (1114–1187) in 1150; Gerard's translation is considered the best and the most widely used.

5. *Kitāb fil-jabr wa'l muqābala* (Book on algebra) by Abū Kamil (circa 850–930), the first major Arabic algebraist after al-Khwārizmī. This has three parts: (i) "On the solution of quadratic equations," (ii) "On applications of algebra to the regular pentagon and decagon," and (iii) "On Diophantine equations and problems of recreational mathematics." Part (ii) involved is an application of algebra to geometric problems and is a combination of Greek methods and those of al-Khwārizmī. Abū Kāmil's book has 74 problems with solutions, and

several of them are also found in *Liber abbaci* (Devlin, page 57). For more on this, see the MacTutor biography of Abu Kamil (accessed 6/11/2023).

"What seems certain is that Leonardo consulted many sources to write *Liber abbaci*, both Latin and Arabic" (Devlin, page 47). We gave biographical information on al-Khwārizmī in Chapter 7, "Chinese, Hindu, and Arabian Mathematics."

Note. As mentioned above, Fibonacci's *Liber abbaci* was not translated into English until 2002. The full reference is: Laurence Sigler, *Fibonacci's Liber Abaci: A Translation into Modern English of Leonardo Pisano's Book of Calculation*, Sources and Studies in the History of Mathematics and Physical Sciences, ed. Gerald Toomer, NY: Springer (2002). The ISBN is 978-0-387-40431-1, and an eBook is available. The Library of Congress call number is QA32.F4713 2002 (the ETSU Sherrod Library has a copy).



Liber abbaci covers the beginnings of algebra and some applied mathematics. It includes many worked examples and exercises. There are fifteen chapters (the titles vary from manuscript to manuscript) which, in Sigler's English translation, are (Devlin, pages 61 and 62):

Dedication and prologue

1. On the recognition of the nine Indian figures and how all numbers are written with them; and how the numbers must be held in the hands, and on the introduction to calculations
2. On the multiplication of whole numbers
3. On the addition of them, one to the other
4. On the subtraction of lesser numbers from greater numbers
5. On the divisions of integral numbers
6. On the multiplication of integral numbers with fractions
7. On the addition and subtraction and division of numbers with fractions and the reduction of several parts to a single part
8. On finding the value of merchandise by the Principal Method
9. On the barter of merchandise and similar things
10. On companies and their members
11. On the alloying of monies
12. On the solutions to many posed problems
13. On the method elchataym and how with it nearly all problems of mathematics are solved

14. On finding square and cubic roots, and on the multiplication, division, and subtraction of them, and on the treatment of binomials and apotomes and their roots

15. On pertinent geometric rules and on problems of algebra and almuchabala.

The first chapter starts: His opening chapter describes how to write—and read—whole numbers in the Hindus' decimal system. Leonardo began (Sigler, page 17): "These are the nine figures of the Indians: 9 8 7 6 5 4 3 2 1. With these nine figures, and with this sign 0 which in Arabic is called zephir, any number whatsoever can be written, as is demonstrated below." He then explains the use of place-value and calculation using the fingers (common among medieval scholars and traders; Devlin, pages 62 and 63)

Note. In **Chapter 2**, Leonardo presents multiplication in much the way we do it today. In **Chapter 3** he covers addition and in **Chapter 4** subtraction. In **Chapter 5** whole number division and simple fractions are described. **Chapters 6 and 7** deal with "mixed numbers." That is, it covers calculations with numbers which have both a hole number part and a fractional part. As today, the mixed numbers are converted to "improper fractions" (in which the numerator is larger is size than the denominator) and computations are performed with these, then the answer is converted back to a mixed number. Worked examples of the previous material are illustrated using problems from business (buying, selling, and pricing) in **Chapters 8 and 9**. **Chapter 10** covers similar topics related to investments and profit sharing. **Chapter 11** involved converting one currency to another ("money changing"). This was particularly useful at the time, since there were many different currencies circulating in the region around Pisa. **Chapter 12** gives 259 worked examples. These are problems explained entirely in words. That is, the examples are in "rhetorical algebra," as opposed to the symbolic algebra which are are used to (and which is first introduced in François Viète's 1591 book *In artem analyticam isagoge* ("Introduction to the analytic art"). This is discussed more in Section 8.9. François Viète. The term *elchataym* in the title of **Chapter 13** ("On the method elchataym and how with it nearly all problems of mathematics are solved") refers to the idea of "double false position." The idea is that two approximations to a desired solution are adjusted to get a better approximation (for example, if one approximation is known to be too low and the other too high, then a midpoint or interpolation could be used to find the better approximation). **Chapter 14** considers roots of numbers. Basing his approach on Euclid's Book X of *Elements*, he addresses sums and differences of unlike roots (that is, square roots of different whole numbers such as $\sqrt{2}$ and $\sqrt{5}$). **Chapter 15** deals with algebraic equations rhetorically, following the methods of al-Khwārizmī's *Algebra* (page 63, 64, 65, and 66).

Note. To illustrate Leonardo's solution technique, we consider one of his "Buying Horses" problems. In Chapter 12, one of the problems is as follows. We give the statement of the problem as given on page 71 of Devlin (it appears in Sigler on page 337, but the statement and solution given by Devlin is more concise):

> "Two men having bezants [a monetary unit] found a horse for sale; as they wished to buy him, the first said to the second, If you will give me 1/3 of your bezants, then I shall have the price of the horse. And the other man proposed to have similarly the price of the horse if he takes 1/4 of the first's bezants. The price of the horse and the bezants of each man are sought."

Leonardo gives his solution rhetorically as:

> "You put 1/4 1/3 in order, and you subtract the 1 which is over the 3 from the 3 itself; there remains 2 that you multiply by the 4; there will be 8 bezants, and the first has this many. Also the 1 which is over the 4 is subtracted from the 4; there remains 3 that you multiply by the 3; there remains 9 bezants, and the other man has this many. Again you multiply the 3 by the 4; there will be 12 from which you take the 1 that comes out of the multiplication of the 1 which is over the 3 by the 1 which is over the 4; there remain 11 bezants for the price of the horse; this method proceeds from the rule of proportion, namely from the finding of the proportion of the bezants of one man to the bezants of the other; the proportion is found thus."

Leonardo has given a very "cookbook" solution that, step-by-step, states what digits are to be written and where, leaving nothing to chance. A symbolic solution is not hard. With x as the number of bezants the first man has, y is the number of bezants the second man has, and p is the price of the horse, we have: $p = x + \frac{1}{3}y$ and $p = \frac{1}{4}x + y$, which implies $\frac{3}{4}x = \frac{2}{3}y$ or $9x = 8y$, so that the (smallest positive) integer solution is $x = 8$, $y = 9$, and $p = 11$.

Note. Now for a quick comment on notation. In Leonardo's time, the Hindu-Arabic numerals were written with fractions before whole numbers (opposite from what we are used to) and fractions were "built up" from right to left, with each new fraction representing that part of the one to its right. To illustrate this, we have that $\frac{1}{2}\,\frac{2}{3}\,\frac{4}{5}$ represents $\frac{1}{2\times 3\times 5}+\frac{2}{3\times 5}+\frac{4}{5}$, which simplifies to $\frac{29}{30}$. (notice the role of reciprocals $1/n$ and how they interact with the fraction to the left and right; 2/3 interacts with 4/5 in such a way that the numerator 4 is ignored, and 1/2 then interacts with the result, 2/15, in such a way that the numerator 2 is ignored). Leonardo, attached to the idea of expressing numbers rhetorically, would have stated this as the Arabic mathematicians would both write and speak it: "Four fifths, and two thirds of a fifth, and one half of a third of a fifth." (Devlin, page 72) Fractions written after the whole number part in Leonardo's time denoted multiplication. For example, 1/2 of 3.14159 would have been written

$$\frac{9}{10}\,\frac{5}{10}\,\frac{1}{10}\,\frac{4}{10}\,\frac{1}{10}\,3\,\frac{1}{2}$$

(Devlin, page 73)

Note. The techniques of solving Leonardo's problems include "the Rule of Three," "the Rule of False Position," and the Rule of Double False Position." The Rule of Three can be applied to certain problems involving proportions. In particular, a problem of the following type:

> Given three numbers, find a fourth in such proportion to the third as the second is to the first.

The Rule of Three then says:

> First place the numbers in such order that the first and third be of one kind, and the second the same as the number required. Multiply the second and third numbers together, and divide the product by the first, the quotient will be the answer to the question.

Symbolically, we would express that as: given numbers a, b, c, find a number x such that x is to c as b is to a. That is, solve for x in the proportions $x : c = b : a$, or $x/c = b/a$. Of course we just multiply both sides by c to get $a = bc/a$ (Devlin, page 74) The Rule of Three was known in China as early as the first century, and Indian texts discussed it from the fifth century onward (Devlin, page 75).

Note. The Rule of False Position involves solving linear equations of the type $Ax = B$. Most of the problems in Chapter 12 of *Liber abbaci* relate to equations of this source (stated rhetorically, of course). After presenting several problems to illustrate the technique and the use of the Hindu-Arabic numbers in this setting, the first application involves a the length of a tree. As stated by Sigler on his page 268, the problem is

> "There is a tree 1/4 1/3 of which lies underground, and it is 21 palms; it is sought what is the length of the tree..."

The expression "1/4 1/3" means "1/4 + 1/3" (expressing fractional parts using sums of reciprocals was common practice at the time (Devlin, page 77). Leonardo's solution starts:

> "...because the least common denominator of 1/4 and 1/3 is 12, you see that the tree is divisible in to 12 equal parts; three plus four parts are 7 parts, and 21 palms; therefore as the 7 is to the 21, so proportionally the 12 is to the length of the tree."

That is, $7/21 = 12/\ell$ where ℓ is the length of the tree (hence $\ell = 36$ palms). Leonardo again rhetorically uses the Rule of False Position in his solution.

Note. In Chapter 13, "The Rule of Double False Position" (or *elchataym*) is illustrated. The technique seems to have its origins in China sometime before 100 C.E. (Devlin, page 78). It can be used to solve linear equations of the form $Ax = B$ (as can the Method of False Position), and also can be used for the more general form $Ax + B = C$. To illustrate this, we consider the problem "On a Man Who Ent on Business to Lucca" from page 460 of Sigler):

> "A certain man went on business to Lucca, next to Florence, and then back to Pisa, and he made double in each city, and in each city he spent 12 denari, and in the end nothing was left for him. It is sought how much he had at the beginning."

Leonardo's solution starts with a guess of 12 denari (the amount he spent in Lucca). By going through the doubling and spending story he finds that man would have 12 denari left (this is "the error"). Since this is too much, Leonardo next guesses that the businessman had 11 denari initially. The story then implies that he then would have 4 denari at the end of the trip. So the second guess has decreased the amount left at the end of the trip by 8 denari (down from 12 to 4). He now argues that he needs to decrease the amount left by *half again* what it decreased by from the first guess (of 12) to the second guess (of 11). That is, he concludes that the correct amount is $10\frac{1}{2}$. You can see the use of a linear equation (changing the guess by 1 unit gave a change in the error of 8 units; this is related to a slope of 1/8 or 8, depending on the choice of the dependent and independent variables). We would solve this by letting A be the amount the man started with, noticing that (1) after Lucca he has $2A - 12$ denari, (2) after Florence he has $2(2A - 12) + 12 = 4A - 36$ denari, and (3) after Pisa he has $2(4A - 36) - 12 = 8A - 84$ denari. Setting the last quantity equal to 0 gives $A = 10\frac{1}{2}$ denari, as expected.

Note. After the last example in Chapter 15, *Liber abbaci* just ends. There is no summary or conclusion. This was common in medieval mathematical texts (Devlin, page 85). Devlin says (on his page 86):

> "The greatness of *Liber abbaci* is due to its quality, its comprehensive nature, and its timeliness: It was good, it provided merchants, bankers, businesspeople, and scholars with everything they needed to know about the new arithmetic methods, and it was the first to do so.

Note. The first edition of *Liber abbaci* in 1202 and the second edition in 1228 (no copies of the first edition survive, so all of our knowledge on this is based on the 1228 edition). This was not Leonardo's only mathematical work. *De practica geometrie* ("Practical geometry") was completed in 1223 and covered surveying and land measurement. It includes mathematical verifications of the methods and uses theorem from Euclid's *Elements* and *On Divisions*. It is in print as *Fibonacci's De Practica Geometrie*, Sources and Studies in the History of Mathematics and Physical Sciences, ed. Barnabas Hughes (Springer, 2008). In 1225 *Flos* ("The flower") was published. It covers some algebra and solutions to problems posed in a contest organized by the Holy Roman Emporer Frederick II. Also published in 1225 is *Liber quadratorum* ("The book of squares"), which covered algebra and number theory and "is Leonardo's most mathematically impressive work" (Devlin, page 88). It is in print as *Leonardo Pisano Fibonacci, The Book of Squares*, An Annotated Translation into Modern English by L.E. Sigler (Academic Press, 1987). The only surviving works of Leonardo are these four books, and an undated letter he wrote to Theodorus Physicus (the imperial philosopher) called *Epistola ad Magistrum Theodorum* discussing three math problems. There does not seem to be an English translation of this, but some of the content is explained in A. F. Horadam, "Fibonacci's Mathematical Letter to Master Theodorus," *Fibonacci Quarterly*, **29**, 103–107 (May 1991), available on the Pennsylvania State University CiteSeerx website (accessed 6/13/2023). Other works by Leonardo are known due to reference by others, but have not survived. This include a book on commercial applications of arithmetic, *Libro di minor guisa*, and a work discussing Book X of Euclid's *Elements* in which he addresses irrational numbers (Devlin, pages 88 and 89).

Note. Holy Roman Emporer Frederick II learned of *Liber abbaci* from his court scholars. He arranged for a meeting with Leonardo at which they discussed *Liber abbaci* and Leonardo demonstrated his mathematical ability. Leonardo write of this in the prologue of *Liber quadratorum* (The book of squares) and dedicates this work to the emporer. This demonstrates that there was recognition of Leonardo and his talent during his lifetime. Rare in a leader of the time (and, arguably, any time) Frederick II had a passion for learning, particularly science and mathematics. In 1224, he founded the university in Naples that bears his name: University of Naples Frederick II, "Universita degli Studi di Napoli Federico II" (Devlin, pages 90 and 91).

Note. Sometime in the last half of the thirteenth century, manuscripts on arithmetic for nonexperts written in Italian started to proliferate. These "abbacus books" may have numbered a thousand by the fifteenth century, about 400 of which have survived. Oddly, these manuscripts were largely ignored by historians of math until Gino Arrighi started publishing studies of them (in Italian) in the 1960s. The content of abbacus books were all similar. They explained the use of the Hindu-Arabic numerals, place values, calculating, and numerous examples (much like *Liber abbaci*; Devlin, pages 103 and 104).

Note. Johannes Gutenberg (circa 1400–February 2, 1468) had his printing press operational by 1450. Before this, all books were hand copied by scribes. The famous Gutenberg Bible was published in Mainz (Germany) in 1454 or 1455. There were 180 copies printed of which 49 complete or almost-complete copies survive. The first printed abbacus book is known today as the *Treviso Arithmetic* and was printed at Treviso, near Venice, in 1478. The author is unknown and when printed, the book had no title. It was written in the common Venetian dialect for a general audience and used commercial problems to explain addition/subtraction and multiplication/division (Devlin, page 106). A few years later in 1494, Luca Pacioli (1445–1517) published *Summa de arithmetica, geometria, proportioni et proportionalità* in Italian (more details on Pacioli are given in Section 8.5. The Fifteenth Century). Pacioli's book considered *negative* numbers, unlike the other abacus books of the time. This was a new idea in Europe and Pacioli is believed to have provided the first printed explanation (Devlin, page 106). *Summa* (as it is commonly abbreviated) was around 600 pages long, concentrated on commercial applications, and is famous (in part) for given one of the earliest explanations of double-entry bookkeeping (Devlin, page 107).

A 1994 Italian stamp commemorating the 500th anniversary of publication of *Summa* from the Wikipedia *Summa de arithmetica* page (accessed 6/13/2023)

Note. Along with the abbacus books came abbacus schools, called *scuole d'abbaco* or *botteghe d'abbaco*. The first record of an abbacus school is one in Verona in 1294. Italian boys typically attended elementary school between the ages of six and eleven. They then could attend a grammar school or an abbacus school. Grammar schools lasted four or five years and covered reading and writing Latin so that students would be prepared for a career as a cleric, notary, lawyer, physician, or grammar school teacher. The abbacus schools lasted two years and covered arithmetic, geometry, and accounting, to prepare students for the business world. Leonardo da Vinci and Niccolò Machiavelli were both taught in abbacus schools. . . " (Devlin, page 108) Between the mid 1300s and 1500, a significant proportion of city dwelling school-age boys attended an abbacus school (in Venice, Milan, Pisa, Siena, Lucca, etc.). In the largest cities, the abbacus school was privately owned and pupils paid fees to attend. More common was an abbacus school formed by the fathers of wealthy merchants that used the school in educate their sons in preparation for working in the family business. Since the abbacus books, like *Liber abbaci*, had solutions following the statement of each problem, they were used as references by teachers and not as textbooks. (Devlin, pages 108, 110, and 111). Abbacus schools spread the Hindu-Arabic numerals and by the late 1400s most Italian merchants had switched from Roman numerals to the new, more efficient system. However, public opposition to change and objections by those trained in the use of a mechanical abacus slowed universal acceptance of new system in Italy (Devlin, page 113). The spread of the Hindu-Arabic numerals outside of Italy was slow, as mentioned in Section 1.9. The Hindu-Arabic Numeral System. Abacus-board arithmetic was still dominant in northern Europe up to the end of the sixteenth century (Devlin, page 115).

Note. *Liber abbaci* remained in manuscript form for centuries and was not printed until 1857, as mentioned above. As a result, it was not widely circulated except to a few dedicated scholars. It became forgotten and was replaced by other shorter, simpler, and more widely available works. The newer works did not reference *Liber abbaci* (nor did they reference each other); citing sources did not become common until much later. One exception was Pacioli's 1494 *Summa*. He explicitly mentions a prominent source: "Since we follow for the most part Leonardo Pisano, I intend to clarify now that any enunciation mentioned without the name of the author is to be attributed to Leonardo." This one reference later lead to the revelation of Leonardo's role in the history of mathematics (Devlin, pages 7 and 8). Pietro

Cossali (June 29, 1748–December 20, 1815) was studying Pacioli's *Summa* while preparing his own two volume book *Origine, transporto in Italia, primi progressi in essa dell-algegra* ("Origins, transmission to Italy, and early progress of algebra there"; Parma 1797, 1799). He found the reference to Leonardo and followed the lead. Cossali concluded that Leonardo's *Liber abbaci* was the principal motivation for the transmission of modern arithmetic and algebra to Italy and, eventually, throughout Europe (Devlin, page 9). The absence of cross references and the small number of passages copied from *Liber abbaci* in the abbacus books made Cossali's claim one requiring more evidence. Historians and archivists spent many years collecting such evidence, and we can now confidently place Leonardo of Pisa in his appropriate historical role as a key figure in the spread of the Hindu-Arabic numerals across Italy and, ultimately, Europe (Devlin, pages 119 and 120).

Note. Before considering the most recent evidence, we review the perspective of math historians during the last half of the 20th century. The majority of such scholars viewed the abbacus books and the abbacus schools as a tradition started by the appearance of *Liber abbaci.* In 1970, Kurt Vogel in his entry on Leonardo for the *Dictionary of Scientific Bilbiography* states: "In surveying Leonardo's activity, one sees him decisively take the role of a pioneet in the revival of mathematics in the Christian West. ...With Leonardo a new epoch in Western mathematics began..." (Devlin, page 129). Warren Van Egmond in his 1980 catalog of abbacus books, he states: "All the manuscripts and books contained in the present catalog can be regarded as members of this tradition [abbacus books] and direct descendants of Leonardo's book. ...The abbacus books thus represent a continuous and remarkably uniform tradition that stretches from the work of Leonardo Pisano to the end of the sixteenth century…" (Devlin, page 129). Well-knonw math historian Ivor Grattan-Guiness states in Volume 1 of the *Companion Encyclopedia of the History and Philosophy of the Mathematical Sciences* (Routledge, 1992): The immediate origin of this tradition lies unquestionably in the Latin *Liver abbaci* ('Book of the Abbacus') written in 1202 by Leonardo of Pisa—more familiar to modern readers by his nickname, Fibonacci" (Devlin, page 130). Laurence Sigler in his 2002 translation of *Liber abbaci* into English, says in his introduction: "For three centuries or so a curriculum based upon Leonardo's *Liber abaci* was taught in Tuscany in schools of abaco normally attended by boys intending to be merchants or by other desiring to learn mathematics" (Devlin, page 130).

Note. We now consider the evidence in support of Leonardo's influence. In 2003, Italian scholar Refaella Franci published results connecting *Liber abbaci* to the abbacus books, and showing that "Leonardo was not just the inspiration for the arithmetic revolution but also the instigator in terms of its content and form" (Devlin, page 121). Her research was based on a manuscript she found in the Biblioteca Riccardiana in Florence. The manuscript, known as *Livero do l'abbecho* ("Book of abbacus"), is undated and the author is unknown. Dates in some of the problems place the manuscript at around 1290 and places mentioned imply the Umbria region of central Italy. The manuscript begins: "This is the book of abacus according to the opinion of master Leonardo of the house of son of Bonaçie from Pisa" (Devlin, page 134). Roughly three-quarters of the problems in the book are translations (into the vernacular of the book) of problems from Chapters 8, 9, 10, and 11 of *Liber abbaci*, including the Rabbit Problem that leads to the

Fibnonacci numbers (Devlin, page 135). However, Chapters 12, 13, and 14 of *Livero do l'abbecho* concern calculating interest and depreciation, topics that are barely mentioned in *Liber abbaci*. In Chapter 14 includes questions on how to handle payments made on different dates; this type of problem is completely absent from *Liber abbaci*. The author of the book shows no particular math skill that would suggest that they could choose such material for inclusion, and it is likely that they simply copied it from another work. Franci suggests that the source for this material is Leonardo's *Libro di merchaanti detto di minor guisa* ("Smaller book for merchants"), which is now lost. "There is no evidence of anyone living at that time other than Leonardo who had the mathematical ability to write an original work of that kind. ...Franci's conclusion has now been supported by examinations of other texts written in the first half of the fourteenth century." (Devlin, page 138). Devlin concludes his thoughts on Leonardo's influence (on his page 140):

> "Taken all together, the evidence is overwhelming. Leonardo left two important legacies: One, comprising his scholarly books *Liber abbaci*, *De practica geometrie*, and *Liber quadratorum*, would lead to the development of modern mathematics. The other, his *Libro di minor guisa*, provided the template for all the abbacus books and the associated growth of practical, commercial arithmetic. Leonardo of Pisa started the modern arithmetic revolution."

Note. To wrap up our discussion of Leonardo of Pisa, we present some of the recognition he has been given. First, as commented above, his name is not "Fibonacci." In the opening statement of *Liber abbaci* it is stated that the book is composed by "*Leonardo Pisani, filius Bonacci.*" The last two Latin words literally translate as "son of Bonacci," but his father's name is Guilielmo. Devlin (page 13) proposes that this should be interpreted as "of the Bonacci family." It was historian Guillaume Libri in 1838 who used the phrase *filius Bonacci* to give Leonardo his nickname of "Fibonacci." The nickname spread in connection with the Fibnonacci sequence and the "Fibonacci numbers," as described in Section 8.3. Fibonacci and the Thirteenth Century. The Fibonacci numbers were given their name by François Édouard Lucas (April 4, 1842–October 3, 1891) in 1870s, following the lead of his countryman Libri. Interest in the Fibonacci numbers inspired the formation of the Fibonacci Society (accessed 6/16/2023) in 1963 and the associated publication of the *Fibonacci Quarterly*.

Note. There seem to be only two physical monuments to Leonardo of Pisa. In 1241, the city of Pisa agreed that Leonardo should be given and annual stipend for his service to the city. The text of the proclamation was engraved on a marble tablet and dedicated on June 16, 1867. It is in the entrance of the State Archives of Pisa (Devlin, page 148).

The marble tablet in the State Archives of Pisa (on the wall to the left). Image from Devlin's Angle blog, October 1, 2012 (accessed 6/17/2023)

The other monument is a statue of Leonardo in the Monumental Cemetery of Pisa in the Piazza dei Miracoli near the Leaning Tower. An inscription on the pedestal containing the statue reads *A Leonardo Fibonacci Insigne Matematico Pisano del Secolo XII* ("To Leonardo Fibonacci, noted mathematician of Pisa of the 12th Century"). There is a strong resemblance of the face of the statue to the drawing from the 1850 *I Benefattori dell'Umani'a* ("The Benefactors of Humanity"), mentioned above, that is likely that the sculpture, Giovanni Paganucci, referenced the drawing.

Image from Wikimedia.org (accessed 6/17/2023)

U.S. and German soldiers fought a month-long battle near the end of World War II that resulted in damage to the statue of Leonardo (amazingly, only minor damage to the hands!). After rebuilding efforts, the statue was put in a public park in 1966 and eventually returned to to the original location in 1990 (after the necessary cleaning it needed from the usual bird "damage" of standing in a public park; Devlin, pages 152 and 153).

Note. Of course the real monument to Leonardo of Pisa is the surviving copies of his work. There are fourteen medieval extant (i.e., surviving) copies of *Liber abbaci*. All are copies of the 1228 second edition. No *original* copies of *Liber abbaci* survive, so these copies are the closest we have to the original. Most of the copies are fragmentary. Three of the copies are complete or almost complete (each of these three are housed in Italy' Devlin, page 153). Devlin concludes with some poignant remarks (pages 157 and 158):

> "History almost forgot him, only a nickname given to him by a later historian surviving, and then only to refer to a sequence of numbers arising from a word problem he copied into his book from another source. Meanwhile, for several centuries his true legacy hung on a single brief reference in one book (Pacioli's *Summa*). ... We live [Leonardo's legacy] every day, every time we do something that depends upon the modern arithmetic he brought to the West."

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