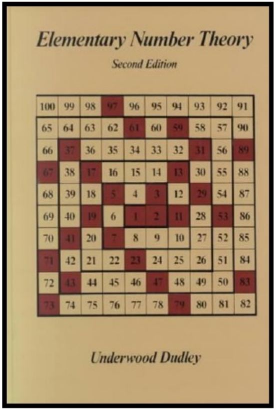
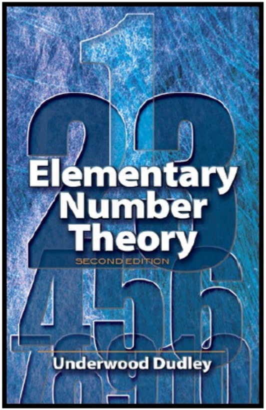
Elementary Number Theory

Section 11. Quadratic Congruences—Proofs of Theorems





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Theorem 11.1

Theorem 11.1. Suppose that p is an odd prime. If $p \nmid a$, then $x^2 \equiv a \pmod{p}$ has exactly two (least residue) solutions or no solutions.

Proof. Notice that $a \not\equiv 0 \pmod{p}$, since $p \nmid a$. Suppose that the congruence has a solution, say r. Then $p - r$ is a solution too, since $(p-r)^2 \equiv p^2 - 2pr + r^2 \equiv r^2 \equiv a \pmod{p}$. If $r = p - r \pmod{p}$, then $2r \equiv 0 \pmod{p}$; but $(2, p) = 1$ so by Theorem 4.4 we have $r \equiv 0 \pmod{p}$, contradicting the fact that $a \not\equiv 0 \pmod{p}$. So r and $p - r$ are different solutions.

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Theorem 11.2. Euler's Criterion

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If p is an odd prime and $p \nmid a$, then $x^2 \equiv a \pmod{p}$ has a solution or no solution depending on whether $a^{(p-1)/2} \equiv 1 \pmod{p}$, or $a^{(p-1)/2} \equiv -1 \pmod{p}$, respectively.

Proof. Let g be a primitive root of the odd prime p (there are $\varphi(p-1)$ primitive roots of p, by Theorem 10.6). Since g is a primitive root of p, then by the definition of "primitive root" it has order $\varphi(p) = p - 1$. Then $a \equiv g^k \pmod{p}$ for some k (since $a \not\equiv 0 \pmod{p}$).

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If k is even, then $x^2 \equiv a \pmod{p}$ has a solution, namely the least residue of $g^{k/2}$. By Fermat's (Little) Theorem, Theorem 6.1, $g^{p-1} \equiv 1 \pmod{p}$, so we have

$$a^{(p-1)/2} \equiv (g^k)^{(p-1)/2} \equiv (g^{k/2})^{(p-1)} \equiv (g^{(p-1)})^{k/2} \equiv 1 \pmod{p}.$$

So $a^{(p-1)/2} \equiv 1 \pmod{p}$ and $x^2 \equiv a \pmod{p}$ has a solution (namely $x = g^{k/2}$), as claimed.

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Proof (continued). If k is odd, then (since $a \equiv g^k \pmod{p}$). Now $x^2 \equiv 1 \pmod{p}$ has two solutions (by Theorem 11.1), namely 1 and $p-1$. We know that $g^{(p-1)} \equiv 1 \pmod{p}$, so $g^{(p-1)/2}$ must have least residue 1 or $p-1$; it can't be 1, or else the order of g is less than $p-1$ and g is not a primitive root of p. So it must be that $g^{(p-1)/2} \equiv p-1 \equiv -1 \pmod{p}$. Therefore $a^{(p-1)/2} \equiv (g^k)^{(p-1)/2} \equiv (g^{(p-1)/2})^k \equiv (-1)^k \equiv -1 \pmod{p}$. Now if $x^2 \equiv a \pmod{p}$ has a solution, say r, then we would have

$$1 \equiv r^{p-1} \equiv (r^2)^{(p-1)/2} \equiv a^{(p-1)/2} \equiv -1 \pmod{p},$$

a contradiction. So there is no solution to $x^2 \equiv a \pmod{p}$. That is, $a^{(p-1)/2} \equiv -1 \pmod{p}$ and $x^2 \equiv a \pmod{p}$ has no solution, as claimed. Since this covers both parities of k, the result holds. □

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Theorem 11.3. The Legendre symbol has the properties

(A) if $a \equiv b \pmod{p}$, then $(a/p) = (b/p)$,

(B) if $p \nmid a$, then $(a^2/p) = 1$, and

(C) if $p \nmid a$ and $p \nmid b$, then $(ab/p) = (a/p)(b/p)$.

Proof. (A) Suppose $(a/p) = 1$, so that $x^2 \equiv a \pmod{p}$ has a solution. Since $a \equiv b \pmod{p}$, then $x^2 \equiv b \pmod{p}$ also has a solution; namely, the same solution as $x^2 \equiv a \pmod{p}$ has. Hence $(b/p) = 1$, as claimed.

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Suppose $(a/p) = -1$, so that $x^2 \equiv a \pmod{p}$ does not have a solution. If $(b/p) = 1$ then $x^2 \equiv b \pmod{p}$ has a solution and, as just shown, this would also be a solution to $x^2 \equiv a \pmod{p}$ since $a \equiv b \pmod{p}$, a contradiction. So it must be that $x^2 \equiv b \pmod{p}$ does not have a solution and so $(b/p) = -1$, as claimed.

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Proof (continued). (B) The least residue of $a \pmod{p}$ is a solution to the equation $x^2 \equiv a^2 \pmod{p}$. Hence $(a^2/p) = 1$, as claimed.

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(C) Notice that the Legendre symbol is only defined for p and odd prime, so we are assuming that property of p. By Euler's Criterion (Theorem 11.2) we have that $(a/p) = 1$ if $a^{(p-1)/2} \equiv 1 \pmod{p}$, and $(a/p) = -1$ if $a^{(p-1)/2} \equiv -1 \pmod{p}$. So $(a/p) \equiv a^{(p-1)/2} \pmod{p}$. Since $(xy)^n \equiv x^n y^n \pmod{p}$ for any x and y, then we have

$$(ab/p) \equiv (ab)^{(p-1)/2} \equiv a^{(p-1)/2} b^{(p-1)/2} \equiv (a/p)(b/p) \pmod{p}.$$

Since the three Legendre symbols in this equivalence are either 1 or -1, the only way the two sides of this congruence can be the same is if the left-had side equals the right-hand side, as claimed. □

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$$(-1/p) = 1 \text{ if } p \equiv 1 \pmod{4}, \text{ and } (-1/p) = -1 \text{ if } p \equiv 3 \pmod{4}.$$

Proof. As observed in the proof of Theorem 11.3, Euler's Criterion (Theorem 11.2) gives $(a/p) \equiv a^{(p-1)/2} \pmod{p}$. So we have

$$(-1/p) \equiv (-1)^{(p-1)/2} \pmod{p}.$$

Since $(p-1)/2$ is even for $p \equiv 1 \pmod{4}$, then in this case we have $(-1/p) = 1$, as claimed. Since $(p-1)/2$ is odd if $p \equiv 3 \pmod{4}$, then in this case we have $(-1/p) = -1$, as claimed. □

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