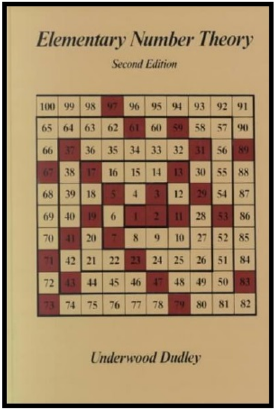
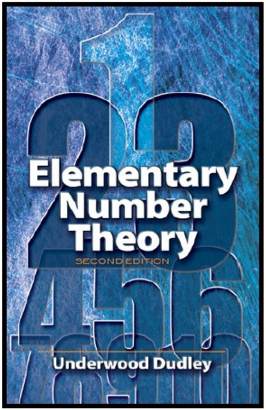
Elementary Number Theory

Section 15. Decimals—Proofs of Theorems





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Theorem 15.1

Theorem 15.1. If a and b are any nonnegative integers, then the decimal expansion of $1/(2^a 5^b)$ terminates.

Proof. Let $M = \max\{a, b\}$. Then $10^M(1/(2^a 5^b)) = 2^{M-a}5^{M-b}$ is an integer, say $n = 2^{M-a}5^{M-b}$. Then $n \leq 10^M$ and so

$$\frac{1}{2^a 5^b} = \frac{2^{M-a}5^{M-b}}{10^M} = \frac{n}{10^M},$$

so the decimal expansion of $1/(2^a 5^b)$ consists of the digits of n, perhaps preceded by some zeros. □

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Theorem 15.2

Theorem 15.2. If $1/n$ has a terminating decimal expansion, then $n = 2^a 5^b$ for some nonnegative integers a and b.

Proof. Let the terminating decimal expansion of $1/n$ be

$$1/n = 0.d_1 d_2 \cdots d_k = \frac{d_1}{10} + \frac{d_2}{10^2} + \cdots + \frac{d_k}{10^k}$$

$$= \frac{d_1 10^{k-1} + d_2 10^{k-2} + \cdots + d_{k-1} 10 + d_k}{10^k}.$$

Let $m = d_1 10^{k-1} + d_2 10^{k-2} + \cdots d_{k-1} 10 + d_k$ (an integer). Then $1/n = m/10^k$ or $mn = 10^k$. The only prime divisors of 10^k are 2 and 5, so the only prime divisors of n are 2 and 5 by Euclid's Lemma (Lemma 2.5). Therefore, n is of the form $2^a 5^b$ for some nonnegative integers a and b, as claimed. □

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Theorem 15.3

Theorem 15.3. The length of the decimal period of $1/n$ is no longer than $n-1$.

Proof. Let t be such that $10^t < n < 10^{t+1}$. Then the Division Algorithm (Theorem 1.2) gives

$$\begin{aligned} 10^{t+1} &= d_1 n + r_1 \text{ where } 0 < r_1 < n, \\ 10 r_1 &= d_2 n + r_2 \text{ where } 0 \le r_2 < n, \\ 10 r_2 &= d_3 n + r_3 \text{ where } 0 \le r_3 < n, \\ &\vdots \\ 10 r_k &= d_{k+1} n + r_{k+1} \text{ where } 0 \le r_{k+1} < n, \\ &\vdots \end{aligned}$$

Now each d_k is less than 10, because for $k \ge 2$ we have $d_k n = 10 r_{k-1} - r_k \le 10 r_{k-1} < 10n$ and $d_1 n = 10^{t+1} - r_1 < 10^{t+1} = 10 \cdot 10^t < 10n$.

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Theorem 15.3 (continued 1)

Proof (continued). The Division Algorithm gave us $10r_k = d_{k+1}n + r_{k+1}$ for $k \geq 1$ above, so dividing by $10n$ on both sides we have

$$r_k/n = d_{k+1}/10 + r_{k+1}/(10n). \qquad (*)$$

We also had $10^{t+1} = d_1 n + r_1$, so dividing both sides by $10^{t+1}n$ gives

$$1/n = d_1/10^{t+1} + r_1/(n10^{t+1}). \qquad (**)$$

Starting with $(**)$ and repeatedly applying $(*)$ gives

$$\begin{aligned} 1/n &= d_1/10^{t+1} + r_1/(n10^{t+1}) \\ &= d_1 10^{t+1} + d_2/10^{t+2} + r_2/(n10^{t+2}) \\ &= d_1/10^{t+1} + d_2/10^{t+2} + d_3/10^{t+3} + r_3/(n10^{t+3}) \\ &\vdots \\ &= d_1/10^{t+1} + d_2/10^{t+2} + d_3/10^{t+3} + d_4/10^{t+4} + \cdots . \end{aligned}$$

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Theorem 15.3. The length of the decimal period of $1/n$ is no longer than $n-1$.

Proof (continued). Therefore the digits in the decimal expansion of $1/n$ has the digits $d_1, d_2, d_3, \ldots$. Each of the remainders $r_1, r_2, \ldots$ is one of the n values $0, 1, 2, \ldots, n-1$. However, if one of the remainders is 0 then the all the decimals from that point on are 0 and the decimal expansion terminates (so that by Theorem 15.2 we must have $n = 2^a 5^b$ for some nonnegative a and b). Hence if the decimal period is not 0 (in which case the claim holds), then the remainders are among the $n-1$ values $1, 2, \ldots, n-1$. So among the n integers $r_1, r_2, \ldots, r_n$ there must be two that are equal (this follows from the Pigeonhole Principle; see my online notes for Mathematical Reasoning [MATH 3000] on Section 4.1. Cardinality; Fundamental Counting Principles). If $r_j = r_k$, then $d_{k+1} = d_{j+1}$, $d_{k+2} = d_{j+2}$, $\ldots$, since d_{k+1} and r_{k+1} are determined by the value of r_k. So the decimal repeats with period no longer than $n-1$, as claimed. □

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Theorem 15.4

Theorem 15.4. If $(n, 10) = 1$, then the period of $1/n$ is r, where r is the smallest positive integer such that $10^r \equiv 1 \pmod{n}$.

Proof. Since $(n, 10) = 1$ then $10^{\varphi}(n) \equiv 1 \pmod{n}$ by Euler's Theorem (Theorem 9.1), so r exists with $10^r \equiv 1 \pmod{n}$. The least residues (mod n) of $1, 10, 10^2, 10^3, \ldots, 10^{n-1}$ may only be the values $1, 2, 3, \ldots, n-1$ because no power of 10 is divisible by n since $(n, 10) = 1$. Now the n residues (mod n) of $1, 10, 10^2, 10^3, \ldots, 10^{n-1}$ can only take on $n-1$ possible values and so, by the Pigeonhole Principle (see my online notes for Mathematical Reasoning [MATH 3000] on Section 4.1. Cardinality; Fundamental Counting Principles), two of the residues are the same.

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Proof (continued). Since $10^r \equiv 1 \pmod{n}$, then $10^r - 1 = kn$ for some integer k. So $k < 10^r$ and in base 10, k has at most r digits. Let

$$k = (d_{r-1}d_{r-2}\cdots d_1 d_0)_{10} = d_{r-1}10^{r-1} + d_{r-2}10^{r-2} + \cdots d_1 10 + d_0,$$

where $0 \leq d_k < 10$ for $k = 0, 1, \ldots, r$. Since $10^r - 1 = kn$ then

$$\begin{aligned}
\frac{1}{n} &= \frac{k}{10^r - 1} = \frac{(d_{r-1}d_{r-2}\cdots d_1 d_0)_{10}}{10^r} \cdot \frac{1}{1 - 10^{-r}} \\
&= (0.d_{r-1}d_{r-2}\cdots d_1 d_0)_{10}(1 + 10^{-r} + 10^{-2r} + 10^{-3r} + \cdots) \\
&\quad \text{since } \frac{1}{1 - 10^{-r}} \text{ is the sum of a geometric series with ratio } 10^{-r} \\
&= 0.\overline{d_{r-1}d_{r-2}\cdots d_1 d_0}.
\end{aligned}$$

So the period of $1/n$ is *at most* r.

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Theorem 15.4. If $(n, 10) = 1$, then the period of $1/n$ is r, where r is the smallest positive integer such that $10^r \equiv 1 \pmod{n}$.

Proof (continued). Suppose the period of $1/n$ is some value s. That is, $1/n = 0.\overline{e_{s-1}e_{s-2}\cdots e_1e_0}$ for some integers $e_0, e_1, \ldots, e_{s-1}$. Then

$$\begin{aligned}\frac{1}{n} &= (0.e_{s-1}e_{s-2}\cdots e_1e_0)(1 + 10^{-s} + 10^{-2s} + \cdots) \\ &= \frac{(e_{s-1}e_{s-2}\cdots e_1e_0)_{10}}{10^s} \cdot \frac{1}{1 - 10^{-s}} \\ &= \frac{(e_{s-1}e_{s-2}\cdots e_1e_0)_{10}}{10^s - 1}.\end{aligned}$$

With $k' = (e_{s-1}e_{s-2}\cdots e_1e_0)_{10}$, we have $nk' = 10^s - 1$, so $10^s \equiv 1 \pmod{n}$. With r as the smallest positive integer such that $10^r \equiv 1 \pmod{n}$, then $s \geq r$. We have that the period of $1/n$ is at most r and at least r, and so the period of $1/n$ is equal to r where r is the smallest positive integer such that $10^r \equiv 1 \pmod{n}$, as claimed. □

Theorem 15.4 (continued 2)

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Proof (continued). Suppose the period of $1/n$ is some value s. That is, $1/n = 0.\overline{e_{s-1}e_{s-2}\cdots e_1 e_0}$ for some integers $e_0, e_1, \ldots, e_{s-1}$. Then

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