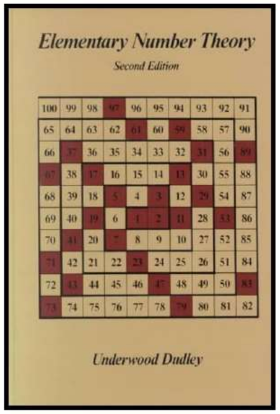
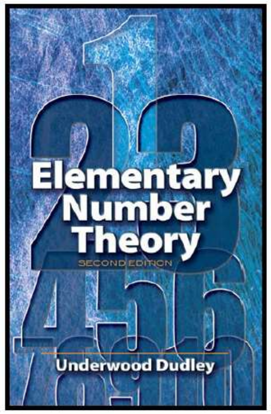
Elementary Number Theory

Section 17. Infinite Descent and Fermat's Conjecture—Proofs of Theorems





Theorem 17.1

Theorem 17.1. There are no nontrivial solutions of $x^4 + y^4 = z^2$.

Proof. ASSUME that a nontrivial solution to $x^4 + y^4 = z^2$ exists. Among the nontrivial solutions, there is one with a smallest value of z^2 (since $z^2 \in \mathbb{N}$; this is part of the definition of $\mathbb{N}$ is a set theoretic setting). Let c^2 denote this value of z^2. Let a and b be corresponding values of x and y, respectively. (Our strategy is to construct $x = r$, $y = s$, $z = t$ that also satisfy $x^2 + y^4 = z^2$ with $t^2 < c^2$, given a contradiction.) Notice that we may suppose that a and b are relatively prime, for if prime p divides a and b then p^2 divides c^2 (by Lemma 1.1) and we have $(a/p)^4 + (b/p)^4 = (c/p^2)^2$, contradicting the minimality of c.

Notice that if a and b are both odd, that is $a \equiv b \equiv 1 \pmod 2$, then $a^2 \equiv b^2 \equiv 1 \pmod 4$ and $a^4 \equiv b^4 \equiv 1 \pmod{16}$. So $a^4 + b^4 \equiv 2 \pmod{16}$. So with $a^4 + b^4 = c^2$ then c must be even, but if $c \equiv 0 \pmod 2$ then $c^2 \equiv 0 \pmod 4 \equiv 2 \pmod{16}$. Hence, we cannot have both a and b odd.

Theorem 17.1 (continued 1)

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Proof (continued). Since a and b are relatively prime, then both cannot be even. So one of a and b is even and the other is odd. Let a be the even one and b the odd one. Then a^2, b^2, c is a fundamental solution of $x^2 + y^2 = z^2$, where $(a^2, b^2) = 1$, a^2 is even, and b^2 is odd. Hence, by Lemma 16.3 there are integers m and n, $m > n$, relatively prime and of opposite parity, such that $a^2 = 2mn$, $b^2 = m^2 - n^2$, and $c = m^2 + n^2$.

We now show that n must be even. ASSUME that n is odd, so that m must be even. Then as mentioned above, $n^2 \equiv 1 \pmod 4$ and $m \equiv 0 \pmod 4$. But then $b^2 = m^2 - n^2 \equiv -1 \pmod 4$. This is a CONTRADICTION because there $x^1 \equiv -1 \pmod 4$ has no solution. So the assumption that n is odd is false, and hence n is even (so that m is odd).

Theorem 17.1 (continued 2)

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Proof (continued). Since n is even, say $n = 2q$, so that $a^2 = 2mn = 4mq$, or $(a/2)^2 = mq$. Next, we show that m and q are relatively prime. ASSUME $(m, q) \neq 1$, say prime $p \mid m$ and $p \mid q$. Then $p \mid 2q$ which means that $p \mid n$. But then prime p divides both m and n, CONTRADICTING the fact that m and n are relatively prime. So the assumption that $(m, q) = 1$ is false, and hence m and q are relatively prime. Therefore, by Lemma 16.2, m and q are both squares, say $m = t^2$ and $q = v^2$. Since $(m, q) = 1$ then $(t^2, v^2) = 1$ and hence $(t, v) = 1$. We saw above that m is odd, so t is also odd.

Since $n^2 + (m^2 - n^2) = m^2$ (D'uh!) then, because $n = 2q = 2v^2$, $m^2 - n^2 = b^2$, and $m = t^2$, we have $(2v^2)2 + b^2 = (t^2)^2$. That is, $(2v^2, b, t^2)$ form a Pythagorean triple.

Theorem 17.1 (continued 3)

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Proof (continued). If $p \mid 2v^2$ and $p \mid b$, then $p \mid n$ (since $n = 2v^2$) and $p \mid b$; and if $p \mid n$ and $p \mid b$, then $p \mid n$ and $p \mid m$ (since $m^2 = b^2 + n^2$). That is, if p divides both $2v^2$ and b, then p divides both n and m. Since $(m, n) = 1$, then there is not p dividing both n and m, and hence there is no p dividing $2v^2$ and b. That is, $(2v^2, b) = 1$. Hence, $2v^2, b, t^2$ is a fundamental solution to $x^2 + y^2 = z^2$ (i.e., $(2v^2, b, t^2)$ is a primitive Pythagorean triple).

By Lemma 16.3, there are integers M and N, with $(M, N) = 1$ and $M \not\equiv N \pmod 2$, such that $2v^2 = 2MN$, $b = M^2 - N^2$, and $t^2 = M^2 + N^2$. So $v^2 = MN$ where $(M, N) = 1$. By Lemma 16.2, we have that $M = r^2$ and $N = s^2$ for some integers r and s. Since $t^2 = M^2 + N^2$, then we have $t^2 = (r^2)^2 + (s^2)^2$, or $r^4 + s^4 = t^2$.

Theorem 17.1 (continued 4)

Theorem 17.1. There are no nontrivial solutions of $x^4 + y^4 = z^2$.

Proof (continued). But then we have another solution of $x^4 + y^4 = z^2$ and in this solution we have $t^2 = m \leq m^2 < m^2 + n^2 = c \leq c^2$. But this is a CONTRADICTION to the fact that c^2 was a minimal value of z^2 among all solutions to $x^4 + y^4 = z^2$. This contradiction shows that the original assumption that there exists a nontrivial solution to $x^4 + y^4 = z^2$ is false. Hence, there are no nontrivial solutions to this equation, as originally claimed. $\square$