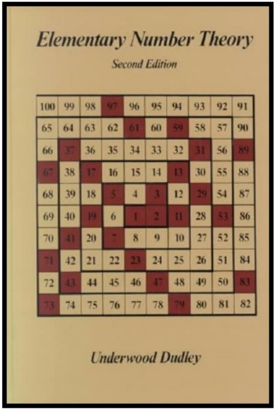
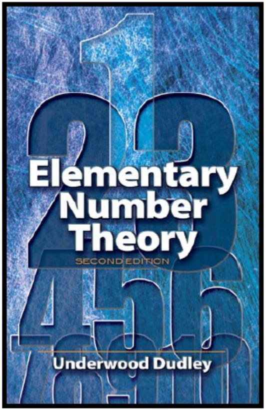
Elementary Number Theory

Section 22. Formulas for Primes—Proofs of Theorems





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Theorem 22.A

Theorem 22.A. An arithmetic progression of prime numbers must be finite in length.

Proof. Suppose the arithmetic progression is given by the function $f(n) = an + b$. Let p be prime and suppose $p \nmid a$, so that $(a, p) = 1$. So by Lemma 5.2 there is (exactly one) integer r such that $ax \equiv -b \pmod{p}$. Then $a(r + kp) + b \equiv ar + b \equiv 0 \pmod{p}$ for all $k \in \{0, 1, 2, \ldots\}$. So every pth term in the sequence is divisible by p (that is, $ar + b$ is divisible by p, $ar + b + p$ is divisible by p, $ar + b + 2p$ is divisible by p, etc.).

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Theorem 22.C

Theorem 22.C. If $f(n) = a_k n^k + a_{k-1} n^{k-1} + \cdots + a_2 n^2 + a_1 n + a_0$ is a polynomial function with integer coefficients, and if r is such that $f(r) \equiv 0 \pmod{p}$ for some p, then $f(r + mp) \equiv f(r) \equiv 0 \pmod{p}$ for all $m \in \mathbb{N}$. That is, no polynomial can have only prime values.

Proof. Notice that we cannot have $f(r) \in \{-1, 0, 1\}$ for all $r \in \mathbb{N}$, unless f is a constant function (and constant functions don't count as polynomial functions). So there is $r \in \mathbb{N}$ such that $f(r) \notin \{-1, 0, 1\}$. For p a prime divisor of such $f(r)$, we have $f(r) \equiv 0 \pmod{p}$.

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$$f(r+mp) = a_k(r+mp)^k + a_{k-1}(r+mp)^{k-1} + \cdots + a_2(r+mp)^2 + a_1(r+mp) + a_0$$

$$\equiv a_k r^k + a_{k-1} r^{k-1} + \cdots + a_2 r^2 + a_1 r + a_0 \equiv f(r) \pmod{p}.$$

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Theorem 22.1

Theorem 22.1. There is a real number θ such that $[\theta^{3^n}]$ is a prime for all $n \in \mathbb{N}$.

Proof. Let p_1 be any prime greater than integer A given in Theorem 2.2.D. Define a sequence of prime numbers recursively for $n = 1, 2, \ldots$ where p_{n+1} satisfies $p_n^3 < p_{n+1} < (p_n + 1)^3 - 1$. Notice that such p_{n+1} always exists by Theorem 2.2.D. Let $u_n = p_n^{3^{-n}}$ and $v_n = (p_n + 1)^{3^{-n}}$ for $n = 1, 2, \ldots$. Since $p_{n+1} > p_n^3$ and 3^{-n-1} is a positive exponent, then $p_{n+1}^{3^{-n-1}} > (p_n^3)^{3^{-n-1}}$ and so as n increases, u_n increases because:

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Theorem 22.1 (continued)

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Proof (continued). Now $u_n = p_n^{3^{-n}} < (p_n + 1)^{3^{-n}} = v_n$, so we now have $u_n < v_n < v_{n-1} < \cdots < v_1$. So $u_n < v_1$ for all $n \in \mathbb{N}$. That is, $\{u_n\}$ is an increasing sequence (and hence nondecreasing) of real numbers that is bounded above by v_1. So by Lemma 22.A, $\{u_n\}$ has a limit, say $\lim_{n\to\infty} u_n = \theta$. Similarly, $v_n > u_n > u_{n-1} > \cdots > u_1$, so we also have $v_n > u_1$ for all $n \in \mathbb{N}$. That is, $\{v_n\}$ is a decreasing sequence (and hence nonincreasing) of real numbers that is bounded below by u_1. So by Lemma 22.B, $\{v_n\}$ has a limit, say $\lim_{n\to\infty} v_n = \varphi$. Since $\{u_n\}$ increases and $\{v_n\}$ decreases, we have $u_n < \theta \le \varphi < v_n$ for all $n \in \mathbb{N}$. Thus $u_n^{3^n} < \theta^{3^n} \le \varphi^{3^n} < v_n^{3^n}$ for all $n \in \mathbb{N}$. Since $u_n^{3^n} = p_n$ and $v_n^{3^n} = p_n + 1$, then we have $p_n < \theta^{3^n} < p_n + 1$. So θ^{3^n} lies between two consecutive *integers*, and hence $[\theta^{3^n}] = p_n$. That is, $[\theta^{3^n}]$ is prime for all $n \in \mathbb{N}$, as claimed. □

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Lemma 22.E

Lemma 22.E. For $n \geq 2$, we have $\prod_{p \leq n} p \leq 2^{2n}$ where p is prime.

Proof. First, observe that by the Binomial Theorem,

$$2^{2m+1} = (1+1)^{2m+1} = 1 + \binom{2m+1}{1} + \binom{2m}{2} + \cdots + \binom{2m+1}{m} + \binom{2m+1}{m+1}$$

$$+ \cdots + \binom{2m+1}{2m} + 1 \geq \binom{2m+1}{m} + \binom{2m+1}{m+1} = 2\binom{2m+1}{m},$$

and so $2^{2m} \geq \binom{2m+1}{m} = \frac{(2m+1)(2M)\cdots(m+2)}{m(m-1)\cdots(2)(1)}$.

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and so $2^{2m} \geq \binom{2m+1}{m} = \dfrac{(2m+1)(2M)\cdots(m+2)}{m(m-1)\cdots(2)(1)}$. Now $\binom{2m+1}{m}$ is divisible by each prime p such that $m+1 < p \leq 2m+1$, so

$$\prod_{m+1<p\leq 2m+1} p \leq \binom{2m+1}{m} \leq 2^{2m}. \qquad (*)$$

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Proof (continued). We now prove the claim by induction. The claim holds for $n = 2$ since $\prod_{p \leq 2} p = 2 \leq 4 = 2^{2(2)}$, so the base case is established. For the induction hypothesis, suppose the claim holds for all $n \leq k$. If k is odd, then $k+1$ is even and

$$\begin{aligned} \prod_{p \leq k+1} p &= \prod_{p \leq k} p \leq 2^{2k} \text{ by the induction hypothesis} \\ &\leq 2^{2(k+1)}, \end{aligned}$$

and the induction step holds when k is odd.

Lemma 22.E (continued 2)

Lemma 22.E. For $n \geq 2$, we have $\prod_{p \leq n} p \leq 2^{2n}$ where p is prime.

Proof (continued). If k is even, say $k = 2m$, then

$$\begin{aligned} \prod_{p \leq k+1} p &= \left(\prod_{p \leq m+1} p \right) \left(\prod_{m+1 < p \leq 2m+1} p \right) \\ &\leq 2^{2(m+1)} 2^{2m} \text{ by the induction hypothesis and } (*) \\ &= 2^{4m+2} = 2^{2(2m+1)} = 2^{2(k+1)}, \end{aligned}$$

and the induction step holds when k is even. So by induction, the claim holds for all $n \geq 2$. □

Lemma 22.F

Lemma 22.F. For $n \geq 1$, we have $\binom{2n}{n} \geq \frac{2^{2n}}{2n}$.

Proof. We prove the claim by induction. For the base case, with $n = 1$ we have $\binom{2n}{n} = \binom{2}{1} = 2 = \frac{4}{2} = \frac{2^{2(1)}}{2(1)} = \frac{2^{2n}}{2n}$. For the induction hypothesis, suppose the claim holds for $n = k$ so that $\binom{2k}{k} \geq \frac{2^{2k}}{2k}$.

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$$\begin{aligned}\binom{2(k+1)}{k+1} &= \binom{2k+2}{k+1} = \frac{(2k+2)!}{(k+1)!(k+1)!} = \frac{(2k+2)(2k+1)(2k)!}{(k+1)^2 k!k!} \\ &= \frac{2(k+1)(2k+1)}{(k+1)^2}\frac{(2k)!}{k!k!} = \frac{2(k+1)(2k+1)}{(k+1)^2}\binom{2k}{k} \\ &\geq \frac{2(k+1)(2k+1)}{(k+1)^2}\frac{2^{2k}}{2k} \text{ by the induction hypothesis}\end{aligned}$$

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\binom{2(k+1)}{k+1} &= \binom{2k+2}{k+1} = \frac{(2k+2)!}{(k+1)!(k+1)!} = \frac{(2k+2)(2k+1)(2k)!}{(k+1)^2 k!k!} \\
&= \frac{2(k+1)(2k+1)}{(k+1)^2}\frac{(2k)!}{k!k!} = \frac{2(k+1)(2k+1)}{(k+1)^2}\binom{2k}{k} \\
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Lemma 22.F (continued)

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so the induction step is established. Hence, the claim holds by induction for all $n \in \mathbb{N}$. □

Theorem 22.2. Bertrand's Theorem

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For all integers $n \geq 2$, there is a prime p such that $n < p < 2n$.

Proof. ASSUME that for some $n \in \mathbb{N}$ there are no primes p such that $n < p < 2n$ or, equivalently, such that $n < p \leq 2n$. For this value of n, let

$$N = \binom{2n}{n} = \frac{(2n)(2n-1)(2n-2)\cdots(n+1)}{n(n-1)(n-2)\cdots(2)(1)}.$$

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So if $2n/3 < p \leq n$, then p is a factor of the denominator, and since $2p > 4n/3 \geq n+1$, then $2p$ is a factor of the numerator. The two p's cancel and, since $3p > 2n$, there are no more factors of p in the numerator. Thus all prime divisors of N are at most $2n/3$, so that by Lemma 22.E

$$\prod_{p \mid N} \leq \prod_{p \leq 2n/3} p \leq 2^{2(2n/3)} = 2^{4n/3}. \qquad (*)$$

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Theorem 22.2. Bertrand's Theorem (continued 1)

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For all integers $n \geq 2$, there is a prime p such that $n < p < 2n$.

Proof (continued). By Lemma 21.4, each prime power in the prime-power decomposition of $N = \binom{2n}{n}$ is at most $2n$. So, if p appears in the prime-power decomposition of N to a power greater than 1, then $p^2 \leq 2n$ (in fact if p^k is in the prime-power decomposition then $p^k \leq 2n$, but we only need the case $k = 2$ since if $p^k \leq 2n$, where $k \geq 2$, then also $p^2 \leq 2n$) and so $p \leq \sqrt{2n}$. There are at most $\sqrt{2n}$ such primes, and since each prime power is at most $2n$, so their total contribution to the prime-power decomposition is at most $(2n)^{\sqrt{2n}}$. All of the other primes appear to the power 1 and, from $(*)$, their product is at most $2^{4n/3}$. That is, the prime divisors of N that appear to the power 1 in the prime-power decomposition of N are bounded by $2^{4n/3}$, and those that appear to a power greater than 1 have a product bounded by $(2n)^{\sqrt{2n}}$.

Theorem 22.2. Bertrand's Theorem (continued 1)

Theorem 22.2. Bertrand's Theorem.
For all integers $n \geq 2$, there is a prime p such that $n < p < 2n$.

Proof (continued). By Lemma 21.4, each prime power in the prime-power decomposition of $N = \binom{2n}{n}$ is at most $2n$. So, if p appears in the prime-power decomposition of N to a power greater than 1, then $p^2 \leq 2n$ (in fact if p^k is in the prime-power decomposition then $p^k \leq 2n$, but we only need the case $k = 2$ since if $p^k \leq 2n$, where $k \geq 2$, then also $p^2 \leq 2n$) and so $p \leq \sqrt{2n}$. There are at most $\sqrt{2n}$ such primes, and since each prime power is at most $2n$, so their total contribution to the prime-power decomposition is at most $(2n)^{\sqrt{2n}}$. All of the other primes appear to the power 1 and, from $(*)$, their product is at most $2^{4n/3}$. That is, the prime divisors of N that appear to the power 1 in the prime-power decomposition of N are bounded by $2^{4n/3}$, and those that appear to a power greater than 1 have a product bounded by $(2n)^{\sqrt{2n}}$.

Theorem 22.2. Bertrand's Theorem (continued 2)

Theorem 22.2. Bertrand's Theorem.
For all integers $n \geq 2$, there is a prime p such that $n < p < 2n$.

Proof (continued). Thus $N = \binom{2n}{n} \leq 2^{4/3}(2n)^{\sqrt{2n}}$. By Lemma 22.F, $\binom{2n}{n} \geq \frac{2^{2n}}{2n}$, so we now have $\frac{2^{2n}}{2n} \leq 2^{4n/3}(2n)^{\sqrt{2n}}$. Taking logarithms of this inequality (remember, the log function is increasing and so preserves inequalities), we get $2n \log 2 - \log 2n \leq (4n/3) \log 2 + \sqrt{2n} \log 2n$, or

$$(2n/3) \log 2 \leq (\sqrt{2n} + 1) \log 2n \leq (\sqrt{2n} + \sqrt{2n}) \log 2n = 2\sqrt{2}\sqrt{n} \log 2n,$$

or $\sqrt{n} \leq \frac{2\sqrt{2} \log 2n}{\log 2}$.

Theorem 22.2. Bertrand's Theorem (continued 3)

Theorem 22.2. Bertrand's Theorem.
For all integers $n \geq 2$, there is a prime p such that $n < p < 2n$.

Proof (continued). $\ldots \sqrt{n} \leq \dfrac{2\sqrt{2}\log 2n}{\log 2}$. But $\sqrt{n}$ increases more rapidly that $\log 2n$, then this inequality if false for n sufficiently large. In fact, we can numerically verify that for $n > 2787$ the inequality is false, and we have CONTRADICTION. So the assumption that there are no primes p such that $n < p < 2n$ is false for $n > 2787$, and so the claim holds provided $n > 2787$. By Note 22.C, we see that the claim holds for $n \leq 9973$, and hence the claim holds for all $n \in \mathbb{N}$, as needed. □

Theorem 22.3

Theorem 22.3. There exists a real number θ such that $[2^\theta]$, $[2^{2^\theta}]$, $[2^{2^{2^\theta}}]$, ... are all prime.

Proof. Let p_1 be any prime, and for $n \in \mathbb{N}$ let p_{n+1} be a prime such that $2^{p_2} < p_{n+1} < 2^{p_n+1}$; notice that such a p_{n+1} exists by Theorem 22.2. Let $u_n = \log^{(n)} p_n$ and $v_n = \log^{(n)} \log^{(n)}(p_n + 1)$, where the function $\log^{(n)}$ is defined recursively as: $\log^{(1)} k = \log_2 k$ and $\log^{(n)} k = \log_2(\log^{(n-1)} k)$. Since $2^{p_2} < p_{n+1} < 2^{p_n+1}$, we have by taking logarithms base 2 that

$$\log_2 2^{p_2} < \log_2 p_{n+1} < \log_2 2^{p_n+1} \text{ or } p_n < \log_2 p_{n+1} < p_n + 1.$$

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$$\log_2 2^{p_2} < \log_2 p_{n+1} < \log_2 2^{p_n+1} \text{ or } p_n < \log_2 p_{n+1} < p_n + 1.$$

Since $p_{n+1} + 1 \leq 2^{p_n+1}$ (because $p_{n+1} < 2^{p_n+1}$) then we have

$$p_n < \log^{(1)} p_{n+1} < \log^{(1)}(p_{n+1} + 1) \leq \log^{(1)}(2^{p_n+1}) = p_n + 1.$$

Taking logarithms base 2 of this inequality n times gives

$$\log^{(n)} p_n < \log^{(n+1)} p_{n+1} < \log^{(n+1)}(p_{n+1} + 1) \leq \log^{(n)}(p_n + 1)$$

or $u_n < u_{n+1} < v_{n+1} \leq v_n$.

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$$\log^{(n)} p_n < \log^{(n+1)} p_{n+1} < \log^{(n+1)}(p_{n+1} + 1) \leq \log^{(n)}(p_n + 1)$$

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Theorem 22.3 (continued)

Theorem 22.3. There exists a real number θ such that $[2^\theta]$, $[2^{2^\theta}]$, $[2^{2^{2^\theta}}]$, ... are all prime.

Proof (continued). ... $u_n < u_{n+1} < v_{n+1} \le v_n$. So sequence $\{u_n\}$ is an increasing (that is, nondecreasing) sequence which is bounded above by v_1, so that it converges by Lemma 22.A, say $\lim_{n\to\infty} u_n = \theta$. Sequence $\{v_n\}$ is a nonincreasing sequence which is bounded below below by u_1, so that it converges by Lemma 22.B, say $\lim_{n\to\infty} v_n = \varphi$. Define $\exp^{(n)} k$ recursively as: $\exp^{(1)} k = 2^k$ and $\exp^{(n)} k = 2^{\exp^{(n+1)} k}$. Since $u_n < \theta < v_n$ for all $n \in \mathbb{N}$, then $\exp^{(n)} u_n < \exp^{(n)} \theta < \exp^{(n)} v_n$, or $p_n < \exp^{(n)} < p_n + 1$. Since $\exp^{(n)} \theta$ lies between two consecutive *integers*, then $[\exp^{(n)} \theta] = p_n$. That is, $[\exp^{(n)} \theta]$ is prime for all $n \in \mathbb{N}$. Since $\exp^{(n)} k$ is defined as an iterated composition of base 2 exponential functions, then we have that each of $[2^\theta]$, $[2^{2^\theta}]$, $[2^{2^{2^\theta}}]$, ... are prime, , as claimed. □

Theorem 22.3 (continued)

Theorem 22.3. There exists a real number θ such that $[2^\theta]$, $[2^{2^\theta}]$, $[2^{2^{2^\theta}}]$, ... are all prime.

Proof (continued). ... $u_n < u_{n+1} < v_{n+1} \leq v_n$. So sequence $\{u_n\}$ is an increasing (that is, nondecreasing) sequence which is bounded above by v_1, so that it converges by Lemma 22.A, say $\lim_{n\to\infty} u_n = \theta$. Sequence $\{v_n\}$ is a nonincreasing sequence which is bounded below below by u_1, so that it converges by Lemma 22.B, say $\lim_{n\to\infty} v_n = \varphi$. Define $\exp^{(n)} k$ recursively as: $\exp^{(1)} k = 2^k$ and $\exp^{(n)} k = 2^{\exp^{(n+1)} k}$. Since $u_n < \theta < v_n$ for all $n \in \mathbb{N}$, then $\exp^{(n)} u_n < \exp^{(n)} \theta < \exp^{(n)} v_n$, or $p_n < \exp^{(n)} < p_n + 1$. Since $\exp^{(n)} \theta$ lies between two consecutive *integers*, then $[\exp^{(n)} \theta] = p_n$. That is, $[\exp^{(n)} \theta]$ is prime for all $n \in \mathbb{N}$. Since $\exp^{(n)} k$ is defined as an iterated composition of base 2 exponential functions, then we have that each of $[2^\theta]$, $[2^{2^\theta}]$, $[2^{2^{2^\theta}}]$, ... are prime, , as claimed. □