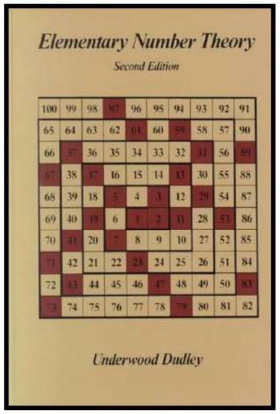
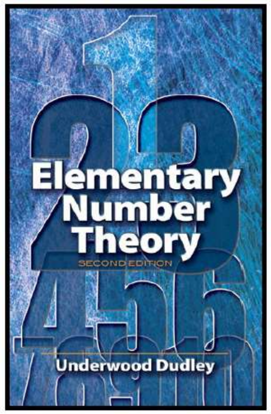
Elementary Number Theory

Section 9. Euler's Theorem and Function—Proofs of Theorems





Lemma 9.1

Lemma 9.1. If $(a, m) = 1$ and $r_1, r_2, \ldots, r_{\varphi(m)}$ are the positive integers less than m and relatively prime to m, then the least residues (mod m) of $ar_1, ar_2, ar_3, \ldots, ar_{\varphi(m)}$ are a permutation of $r_1, r_2, r_3, \ldots, r_{\varphi(m)}$.

Proof. There are exactly $\varphi(m)$ numbers in the collection $ar_1, ar_2, \ldots, ar_{\varphi(m)}$. Since there are also $\varphi(m)$ positive integers less than m that are relatively prime to m, namely $r_1, r_2, \ldots, r_{\varphi(m)}$, we just need to show that the least residues (mod m) of $ar_1, ar_2, \ldots, ar_{\varphi(m)}$ are distinct and are relatively prime to m.

To show that the least residues (mod m) are all different, suppose that some two of them are equal, say $ar_i \equiv ar_j \pmod{m}$ for some $1 \le i \le \varphi(m)$ and $1 \le j \le \varphi(m)$. Since $(a, m) = 1$ then $ar_i \equiv ar_j \pmod{m}$ implies that $r_i \equiv r_j \pmod{m}$ by Theorem 4.4. Since r_i and r_j are least residues (mod m), we have $r_i = r_j$. We have shown that $ar_i \equiv ar_j \pmod{m}$ implies $r_i \neq r_j$. The contrapositive of this is that $r_i = r_j$ implies $ar_i \not\equiv ar_j$. So the numbers $ar_1, ar_2, \ldots, ar_{\varphi(m)}$ are distinct, as claimed.

Lemma 9.1 (continued)

Lemma 9.1. If $(a, m) = 1$ and $r_1, r_2, \ldots, r_{\varphi(m)}$ are the positive integers less than m and relatively prime to m, then the least residues (mod m) of $ar_1, ar_2, ar_3, \ldots, ar_{\varphi(m)}$ are a permutation of $r_1, r_2, r_3, \ldots, r_{\varphi(m)}$.

Proof (continued). Now we show that each of $ar_1, ar_2, \ldots, ar_{\varphi(m)}$ is relatively prime to m. ASSUME that p is a prime common divisor of ar_i and m for some i, where $1 \le i \le \varphi(m)$. Since p is prime then either $p \mid a$ or $[\mid r_i$ by Euclid's Lemma (Lemma 2.5). So either p is a common divisor of a and m, or p is a common divisor of r_i and m. But $(a, m) = (r_i, m) = 1$ by hypothesis so this is a CONTRADICTION. So there is no common divisor of ar_i and m and hence $(ar_i, m) = 1$ for all $i = 1, 2, \ldots, \varphi(m)$, as claimed. □

Theorem 9.1. Euler's Theorem

Theorem 9.1. Euler's Theorem. Suppose that $m \ge 1$ and $(a, m) = 1$. Then $a^{\varphi(m)} \equiv 1 \pmod{m}$.

Proof. By Lemma 9.1 we have

$$r_1 r_2 \cdots r_{\varphi(m)} \equiv (ar_1)(ar_2)\cdots(ar_{\varphi(m)}) \equiv a^{\varphi(m)}(r_1 r_2 \cdots r_{\varphi(m)}) \pmod{m}.$$

Since each of $r_1, r_2, \ldots, r_{\varphi(m)}$ is relatively prime to m, then the product $r_1 r_2 \cdots r_{\varphi(m)}$ is also relatively prime to m (by, for example, the contrapositive of Corollary 1.1 and induction). So by Theorem 4.4, we can cancel $r_1 r_2 \cdots r_{\varphi(m)}$ in the congruence above to get $1 \equiv a^{\varphi(m)} \pmod{m}$, as claimed. □

Lemma 9.2

Lemma 9.2. For prime p, $\varphi(p^n) = p^{n-1}(p-1)$ for all positive integers n.

Proof. The positive integers less that or equal to p^n which are *not* relatively prime to p^n are exactly the multiples of p: $p, 2p, 3p, \ldots, (p^n - 1)p$. This includes p^{n-1} such numbers. There are p^n positive integers less than or equal to p^n, we we have $\varphi(p^n) = p^n - p^{n-1} = p^{n-1}(p-1)$, as claimed. □

Lemma 9.3

Lemma 9.3. If $(a, m) = 1$ and $a \equiv b \pmod{m}$, then $(b, m) = 1$.

Proof. Since $a \equiv b \pmod{m}$ then $b = a + km$ for some positive integer k. Then by Lemma 1.3 (with a, b, r of Lemma 1.3 as b, m, a) we have $(b, m) = (a, m) = 1$, as claimed. □

Corollary 9.A

Corollary 9.A. If the least residues modulo m of $r_1, r_2, \ldots, r_m$ are a permutation of $0, 1, \ldots, m-1$, then the list $r_1, r_2, \ldots, r_m$ contains exactly $\varphi(m)$ elements relatively prime to m.

Proof. First, the least residue of r_i modulo m is some j where $0 \le j \le m-1$; that is $j \equiv r_i \pmod{m}$. If $(j, m) = 1$ then by Lemma 9.3 we have $(r_i, m) = 1$, so that for j relatively prime to m we have r_i relatively prime to m. Conversely, if $(j, m) = d > 1$ then $d \mid j$ and $d \mid m$, so that $d \mid (km + j)$ for every integer k, by Lemma 1.2. Since $j \equiv r_i \pmod{m}$ then $r_i = km + j$ for some integer k and so $d \mid r_i$. That is, if j and m are not relatively prime, then r_i and m are not relatively prime. So r_i is relatively prime to m if and only if j is relatively prime to m. Therefore, since the list $0, 1, \ldots, m-1$ contains exactly $\varphi(m)$ elements relatively prime to m, then the list $r_1, r_2, \ldots, r_m$ contains exactly $\varphi(m)$ elements relatively prime to m, as calimed. □

Theorem 9.2

Theorem 9.2. Euler's φ-function is multiplicative.

Proof. Suppose $(m, n) = 1$. Then consider the numbers from 1 to mn written consecutively in columns as:

$$\begin{array}{ccccc} 1 & m+1 & 2m+1 & \cdots & (n-1)m+1 \\ 2 & m+2 & 2m+2 & \cdots & (n-1)m+2 \\ 3 & m+3 & 2m+3 & \cdots & (n-1)m+3 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ m & 2m & 3m & \cdots & mn. \end{array}$$

Suppose $(m, r) = d$ where $d > 1$. Since $d \mid m$ and $d \mid r$ then by Lemma 1.2 $d \mid (km + r)$ for any nonnegative integer k. Notice that the rows in the array are of the form $r \;\; m+r \;\; 2m+r \;\; \cdots \;\; km+r \;\; \cdots \;\; (n-1)m+r$. So if $d > 1$ divides m and r, then d divides every entry in row r. Hence, any positive number relatively prime to mn (and less than mn) must appear in the array above in a row for which the number is relatively prime to the *first entry* in that row.

Theorem 9.2 (continued 1)

Proof (continued). Now the numbers in the rth row of the array are $r\ \ m+r\ \ 2m+r\ \ \cdots\ \ km+r\ \ \cdots\ \ (n-1)m+r$. We first claim that when r and m are relatively prime, these numbers have least residues modulo n of some permutation of $0, 1, 2, \ldots, (n-1)$. To verify this, it is sufficient to show that no two of the numbers in the rth row are congruent modulo n. Suppose $km + r \equiv jm + r \pmod{n}$ with $0 \le k < n$ and $0 \le j < n$. Then $km \equiv jm \pmod{n}$, and since $(m, n) = 1$ (by our initial hypothesis in the proof) then we have $k \equiv j \pmod{n}$ by Theorem 4.4. Since both k and j are between 0 and $n-1$, then we must have $k = j$. That is, with $0 \le k < n$ and $0 \le j < n$, if $km + r \equiv jm + r \pmod{n}$ then $k = j$. The contrapositive of this result is that (with $0 \le k < n$ and $0 \le j < n$) if $k \ne j$ then $km + r \not\equiv jm + r \pmod{n}$. Therefore no two elements of the rth row are congruent modulo n and hence the least residues modulo n of the numbers in the rth row are some permutation of $0, 1, 2, \ldots, (n-1)$, as claimed.

Theorem 9.2 (continued 2)

Proof (continued). Since the least residues modulo n of the numbers in the rth row are some permutation of $0, 1, 2, \ldots, (n-1)$, then by Corollary 9.A we have that the rth row of the array (when r and m are relatively prime) contains exactly $\varphi(n)$ elements relatively prime to n. By Lemma 9.3 (where r and m are relatively prime), every element in the rth row of the array, $r\ \ m+r\ \ 2m+r\ \ \cdots\ \ km+r\ \ \cdots\ \ (n-1)m+r$, is relatively prime to m. Such an rth row contains exactly $\varphi(n)$ elements that are relatively prime to both m and n, and hence are relatively prime to mn (this follows, say, from Euclid's Lemma [Lemma 2.5] which states that if prime p divides ab then either $p \mid a$ or $p \mid b$). We have seen that a positive number relatively prime to mn (and less than mn) appears in the rth row only when r and m are relatively prime (there are $\varphi(m)$ such rows), and each such row contains $\varphi(n)$ entries relatively prime to mn. So the array contains $\varphi(m)\varphi(n)$ elements relatively prime to mn. That is, $\varphi(mn) = \varphi(m)\varphi(n)$ and φ is multiplicative, as claimed. □

Theorem 9.3

Theorem 9.3. If n has a prime-power decomposition given by $n = p_1^{e_1} p_2^{e_2} \cdots p_k^{e_k}$, then $\varphi(n) = p_1^{e_1-1}(p_1 - 1)p_2^{e_2-1}(p_2 - 1) \cdots p_k^{e_k-1}(p_k - 1)$.

Proof. Since φ is multiplicative, then Theorem 7.5 implies that

$$\varphi(n) = \varphi(p_1^{e_1} p_2^{e_2} \cdots p_k^{e_k}) = \varphi(p_1^{e_1})\varphi(p_2^{e_2}) \cdots \varphi(p_k^{e_k}).$$

By Lemma 9.2, $\varphi(p_i^{e_i}) = p_i^{e_i-1}(p_i - 1)$ for each $1 \le i \le k$, and the claim follows. □

Theorem 9.4

Theorem 9.4. If $n \ge 1$, then $\displaystyle\sum_{d \mid n} \varphi(d) = n$.

Proof. Let positive integer n be given. For the set of integers $S = \{1, 2, \ldots, n\}$, define the set C_d (where $1 \le d \le n$) to consist of those numbers in S that have greatest common divisor with n or d. That is, for given n we have $m \in C_d$ if and only if $(m, n) = d$. But $(m, n) = d$ if and only if $(m/d, n/d) = 1$ by Theorem 1.1. So $m \in C_d$ if and only if m/d is relatively prime to n/d. The number of positive integers less than or equal to n/d and relatively prime to n/d is, by definition, $\varphi(n/d)$. So the number of elements in C_d is $\varphi(n/d)$. Since each element of $S = \{1, 2, \ldots, n\}$ is in exactly one C_d, then $n = \sum_{d \mid n} \varphi(n/d)$. Now if $d \mid n$, then $n = dc$ for some c where $c \mid n$ (and $c = n/d$). So summing $\varphi(n/d)$ over all $d \mid n$, is equivalent to summing $\varphi(c)$ over all $c \mid n$. That is, $\sum_{d \mid n} \varphi(n/d) = \sum_{c \mid n} \varphi(c)$. So $n = \sum_{d \mid n} \varphi(n/d) = \sum_{d \mid n} \varphi(d)$, as claimed. □