Section 13. Numbers in Other Bases

Note. Sections 13, 14, and 15 are on a common theme. In this section we consider representing numbers in bases other than 10, with an emphasis on base 2 representations. The theme continues in the Section 14 where we consider base 12 representations ("duodecimals"). In Section 15 we consider properties of decimal representations of real numbers, such as termination and repetition of these expansions.

Note. We automatically use our base 10 representation of real numbers with little thought. To start the discussion, notice that when we write a real number in decimal form, say $d_k d_{k-1} \cdots d_2 d_1 d_0 \,.\, d_{-1} d_{-2} \cdots$, we mean the real number $\sum_{n=-\infty}^{k} d_n \cdot 10^n$ or, if you like, $\sum_{n=-k}^{\infty} d_{-n} \cdot 10^{-n}$; here we allow for the d_n to be 0, maybe even an infinite number of them. In this section, we consider representations or natural numbers, so we do not have a need to address series and infinite decimal expansions. In the next two theorems we show that each natural number can be written similarly in base 2.

Theorem 13.1. Every positive integer can be written as a sum of distinct powers of 2.

Theorem 13.2. Every positive integer can be written as the sum of the distinct powers of 2 in only one way.

Note. Theorems 13.1 and 13.2 combine to prove that every positive integer n can be written in exactly one way in the form

$$n = d_0 + d_1 \cdot 2 + d_2 \cdot 2^2 + d_3 \cdot 2^3 + \cdot + d_k \cdot 2^k$$

for some $k \in \mathbb{N}$ where each d_i is either 0 or 1. In this case, we write $(d_k d_{k-1} \cdots d_2 d_1 d_0)_2$. This is the *base* 2 *representation* of n.

Exercises 13.4 and 13.5. We can convert base 2 representations into base 10. For example, we have $1001_2 = 1 \cdot 2^3 + 0 \cdot 2^2 + 0 \cdot 2^1 + 1 \cdot 2^0 = 8 + 1 = 9$, $111_2 = 1 \cdot 2^2 + 1 \cdot 2^1 + 1 \cdot 2^0 = 4 + 2 + 1 = 7$, and $1000000_2 = 1 \cdot 2^6 = 64$. We can convert base 10 representations into base 2. For example, we have $2 = 1 \cdot 2^1 + 0 \cdot 2^0 = 10_2$, $20 = 16 + 4 = 1 \cdot 2^4 + 0 \cdot 2^3 + 1 \cdot 2^2 + 0 \cdot 2^1 + 0 \cdot 2^0 = 10100_2$, and $200 = 128 + 64 + 8 = 1 \cdot 2^7 + 1 \cdot 2^6 + 0 \cdot 2^5 + 0 \cdot 2^4 + 1 \cdot 2^3 + 0 \cdot 2^2 + 0 \cdot 2^1 + 0 \cdot 2^0 = 11001000_2$.

Note. In fact, we can use any integer $b \geq 2$ as a base for representations of natural numbers, as we now prove.

Theorem 13.3. Let $b \geq 2$ be any integer (called the *base*). Any positive integer n can be written uniquely in the base b; that is, in the form

$$n = d_0 + d_1 \cdot b + d_2 \cdot b^2 + \cdots + d_k \cdot b^k$$

for some k, with $0 \leq d_i < b$ for $i \in \{0, 1, 2, \ldots, k\}$.

Note. For $b \geq 2$ with $n = d_0 + d_1 b + d_2 b^2 + d_3 b^3 + \cdots + d_k b^k$ (as established in Theorem 13.3), we write $n = (d_k d_{k-1} \cdots d_2 d_1 d_0)_b$.

Note. Notice that Exercises 13.11 and 13.12 address decimal representations to bases other than 10. Exercise 13.17 considers representations base 16; this requires the addition of six new digits (A, B, C, D, E, F).

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