Section 14. Duodecimals

Note. In this section we consider arithmetic using base 12 representations. Through several examples, we illustrate addition, subtraction, multiplication, and division in this setting. There is no new theory in this section and it may be skipped without serious loss.

Note/Definition. We need b symbols (or "digits") for base b representations, since for n a positive integer we have by Theorem 13.3 that

$$n = d_0 + d_1 \cdot b + d_2 \cdot b^2 + \cdots + d_k \cdot b^k = (d_k d_{k-1} \cdots d_2 d_1 d_0)_b$$

where $0 \le d_i < b$ for $i \in \{0, 1, 2, \ldots, k\}$. So for $b = 12$, we use the usual digits 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, along with the symbols χ and ε (the Greek letters chi and epsilon, respectively). These digits form the *duodecimal* system. We then count base 12, we then have

$$1, 2, 3, 4, 5, 6, 7, 8, 9, \chi, \varepsilon, 10, 11, 12, \ldots, 19, 1\chi, 1\varepsilon, 20, \ldots,$$

$$30, 31, \ldots, 40, 41, \ldots, 99, \chi 0, \chi 1, \chi 2, \ldots, \chi 9, \chi\chi, \chi\varepsilon, \varepsilon 0, \varepsilon 1, \ldots, \varepsilon\varepsilon, 100, \ldots.$$

Note. Addition of duodecimals is the same as base 10 addition, we just "carry a 1" when a sum reaches 12. For example, we have the sums:

$$\begin{array}{cccccccccccc} 6 & 6 & 6 & 6 & 6 & 6 & 6 & 6 & 6 & 6 & 6 & 6 \\ \underline{+1} & \underline{+2} & \underline{+3} & \underline{+4} & \underline{+5} & \underline{+6} & \underline{+7} & \underline{+8} & \underline{+9} & \underline{+\chi} & \underline{+\varepsilon} & \underline{+10} \\ 7 & 8 & 9 & \chi & \varepsilon & 10 & 11 & 12 & 13 & 14 & 15 & 16 \end{array}$$

Notice that in this section we do not write the base 12, as we did in the precious section. So when we write "6+10 = 16," we mean $6_{12}+10_{12} = 16_{12}$, this corresponds to $6_{10} + 12_{10} = 18_{10}$.

Note. We can easily produce an addition table of duodecimals. A multiplication table is more challenging, but we can confirm the following.

Duodecimal Multiplication Table

$\cdot$	2	3	4	5	6	7	8	9	χ	ε	10
2	4	6	8	χ	10	12	14	16	18	1χ	20
3	6	9	10	13	16	19	20	23	26	29	30
4	8	10	14	18	20	24	28	30	34	38	40
5	χ	13	18	21	26	2ε	34	39	42	47	50
6	10	16	20	26	30	36	40	46	50	56	60
7	12	19	24	2ε	36	41	48	53	5χ	65	70
8	14	20	28	34	40	48	54	60	68	74	80
9	16	23	30	39	46	53	60	69	76	83	90
χ	18	26	34	42	50	5χ	68	76	84	92	$\chi 0$
ε	1ε	29	38	47	56	65	74	83	92	$\chi 1$	$\varepsilon 0$
10	20	30	40	50	60	70	80	90	$\chi 0$	$\varepsilon 0$	100

Note. The multiplication table allows us to perform more advanced multiplication in the usual way:

$$\begin{array}{r} 34 \\ \underline{\times 5} \\ 18 \\ \underline{+13} \\ 148 \end{array} \qquad \begin{array}{r} 1755 \\ \underline{\times \chi} \\ 42 \\ 42 \\ 5\chi \\ \underline{+\chi} \\ 14262 \end{array} \qquad \begin{array}{r} \chi\chi \\ \underline{\times \varepsilon\varepsilon} \\ 92 \\ 92 \\ 92 \\ \underline{+92} \\ \chi 912 \end{array}$$

Note. Though trickier, we can also use the multiplication table to perform division. As examples, we have:

$$\begin{array}{r} \chi\ 8 \\ 5\,\overline{)\ 4\ 5\ 6} \\ \underline{-\ 4\ 2} \\ 3\ 6 \\ \underline{-\ 3\ 4} \\ 2 \end{array} \qquad \begin{array}{r} 2\ 0 \\ 22\,\overline{)\ 4\ 5\ 6} \\ \underline{-\ 4\ 4} \\ 1\ 6 \end{array} \qquad \begin{array}{r} 1\ 4\ 0 \\ 31\,\overline{)\ 4\ 1\ 5\ 9} \\ \underline{-\ 3\ 1} \\ 1\ 0\ 5 \\ \underline{-\ 1\ 0\ 4} \\ 1\ 9 \end{array}$$

That is, in duodecimals we have that 456/5 has quotient $\chi 8$ with remainder 2, 456/22 has quotient 20 with remainder 16, and 4159/31 has quotient 140 with remainder 19. Notice that the mechanics for multiplication and division are the same in duodecimals as in the base 10 case with which we are familiar.

Note. In the next section we will consider decimal representations of quotients. Sometimes such representations terminate and sometimes they repeat (when dealing with irrational numbers, the decimal representation neither terminates nor repeats). We represent the repeating part with an overline. For example, base 10 we have $1/7 = 0.142857142857142857\cdots = 0.\overline{142857}$. We find the following duodecimal representations of reciprocals:

n	1	2	3	4	5	6	7	8	9	χ	ε
$1/n$	1	$0,6$	0.4	0.3	$0.\overline{2497}$	0.2	$0.\overline{186\chi35}$	0.16	0.14	$0.1\overline{2497}$	$0.\overline{1}$

Revised: 3/16/2022