Section 15. Decimals

Note. In this section we return to standard base 10 representations of real numbers. We consider expressing rational numbers as decimals and determine, for those with a repeating part, how long the repeating part may be.

Note. We denote the real number

$$\frac{d_1}{10} + \frac{d_2}{10^2} + \frac{d_3}{10^3} + \cdots = \sum_{k=1}^{\infty} \frac{d_k}{10^k}$$

where $0 \le d_k < 10$ for all k as $0.d_1d_2d_3\cdots$. A bar over part of a decimal will indicate that this part repeats indefinitely time after time starting with the overlines part (and this is all that appears in the decimal expansion after the bar is introduced). For example, $0.01\overline{47} = 0.01474747\cdots$ and $0.\overline{9} = 0.999\cdots$. We can convert a decimal representation with a repeating pattern into a rational number as follows.

Example 15.A. Let $x = 0.01\overline{47}$. Then $100x = 1.\overline{47}$, so that

$$99x = 100x - x = 1.47\overline{47} - 0.01\overline{47} = 1.46.$$

Therefore, $x = 1.46/99 = 146/9900$ and we have that $0.01\overline{47} = 146/9900$. Also, with $x = 0.\overline{9}$ we have $10x = 9.\overline{9}$ and $9x = 10x - x = 9.\overline{9} - 0.\overline{9} = 9$, so that $x = 1$. Notice that this means that some numbers have two different decimal representations, since we have $1.0 = 0.\overline{9}$. We can similarly take any terminating decimal representation and convert it into an infinite repeating decimal representation. For example, we can write 0.123 as $1.122\overline{9}$.

Note. The next table gives the repeating part of the reciprocal of the integers 2 through 29, and the length of the repeating part. It suggests a pattern that we will establish below in Theorem 15.3.

n	$1/n$	Period	n	$1/n$	Period
2	0.5	0	16	0.0625	0
3	$0.\overline{3}$	0	17	$0.\overline{0588235294117647}$	16
4	0.25	1	18	$0.0\overline{5}$	1
5	0.2	0	19	$0.\overline{052631578947368421}$	18
6	$0.1\overline{6}$	0	20	0.05	0
7	$0.\overline{142857}$	6	21	0.047619	6
8	0.125	0	22	$0.0\overline{45}$	2
9	$0.\overline{1}$	1	23	$0.\overline{0434782608695652173913}$	22
10	0.1	0	24	$0.041\overline{6}$	1
11	$0.\overline{09}$	2	25	0.04	0
12	$0.08\overline{3}$	1	26	$0.0\overline{384615}$	6
13	$0.\overline{076923}$	6	27	$0.\overline{037}$	3
14	$0.0\overline{714285}$	6	28	$0.03\overline{571428}$	6
15	$0.0\overline{6}$	1	29	$0.\overline{0344827586206896551724137931}$	28

We see that the integers with reciprocals that have terminating decimal representations are 2, 4, 5, 8, 10, 16, 20, and 25. Each of these numbers is of the form $2^a 5^b$ for and nonnegative integers a and b. This is suggestive, since base 10 decimal representations have the property that $10 = 2 \cdot 5$. In fact, this is not a coincidence as we now show.

Theorem 15.1. If a and b are any nonnegative integers, then the decimal expansion of $1/(2^a5^b)$ terminates.

Note. The converse of Theorem 15.1 also holds, as we now show.

Theorem 15.2. If $1/n$ has a terminating decimal expansion, then $n = 2^a5^b$ for some nonnegative integers a and b.

Note. We see in the table above that length of the repeating part of the reciprocal of a positive integer n appears to be bounded by $n-1$; this is particularly suggested by some of the prime values of n. We now prove that this holds in general.

Theorem 15.3. The length of the decimal period of $1/n$ is no longer than $n-1$.

Exercise 15.5. We now illustrate the proof of Theorem 15.3 by applying the division algorithm to find the decimal expansion of $1/41$.

Solution. First, we have $10^1 < 41 < 10^2$, so we take $t = 1$ and then we get

$$\begin{aligned} 10^2 &= 2 \cdot 41 + 18 \\ 10 \cdot 18 &= 4 \cdot 41 + 16 \\ 10 \cdot 16 &= 3 \cdot 41 + 37 \\ 10 \cdot 37 &= 9 \cdot 41 + 1 \\ 10 \cdot 1 &= 0 \cdot 41 + 10 \end{aligned}$$

$$10 \cdot 10 \;=\; 2 \cdot 41 + 18.$$

Now we see that the remainders will start to repeat at this stage. Therefore we have

$$\frac{1}{n} = \frac{d_1}{10^{t+1}} + \frac{d_2}{10^{t+2}} + \frac{d_3}{10^{t+3}} + \cdots = 0.0\overline{24390}. \qquad \square$$

Note. We can refine Theorem 15.3 when n is relatively prime to 10.

Theorem 15.4. If $(n, 10) = 1$, then the period of $1/n$ is r, where r is the smallest positive integer such that $10^r \equiv 1 \pmod{n}$.

Note. We now find the period of 1/21. Let $n = 21$. We consider $10^r \pmod{21}$ for small values of r:

r	1	2	3	4	5	6
$10^r \pmod{21}$	10	16	13	4	19	1

So $10^6 - 1 = 999999$ is divisible by 21, and in fact $999999 = 21 \cdot 47619$. So, as in the proof of Theorem 15.4,

$$\frac{1}{21} = \frac{47619}{999999} = \frac{47619}{1000000}\left(1 - \frac{1}{1 - 0.000001}\right)$$
$$= (0.047619)(1 + (0.000001) + (0.000001)^2 + \cdots) = 0.\overline{047619}.$$

Note. So far we have only considered repeated patterns of reciprocals $1/n$. The proof of Theorem 15.4 carries over to general rationals c/n where $(c, n) = 1$ (that is, where c/n is in reduced form). Also, if a fraction is divided by 2 or 5 then its period is unchanged. In summary, we have the next result.

Theorem 15.5. If $n \neq 2^a 5^b$ and $(c, n) = 1$, then the period of the decimal expansion of c/n is r, the smallest positive integer such that $10^r \equiv 1 \pmod{n_1}$, where $n = 2^a 5^b n_1$ and $(n_1, 10) = 1$.

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