# Section 16. Pythagorean Triangles

**Note.** In this section we consider right triangles with integer-length sides (i.e., Pythagorean triangles). We classify all triples of representing these lengths in Theorem 16.1.

**Note.** The Babylonians knew a number of several right triangles with integer-length sides. Such a triple of (positive) integers is called a "Pythagorean triple." The most familiar examples are $3, 4, 5$ and $5, 12, 13$ (since $3^2 + 4^2 = 5^2$ and $5^2 + 12^2 = 13^4$, respectively). Dudley says (page 127) that the Babylonians knew of the Pythagorean triples $(319, 360, 481)$, $(1679, 2400, 2929)$, $(1771, 2700, 3229)$, $(4601, 4900, 6649)$, and $(4961, 6480, 8161)$, and claims that they likely used these as a table of trigonometric function values. It seems that the Babylonians *knew* the Pythagorean Theorem, but they knew of it experimentally instead of in terms of a proof (a proof is thought to originally be given by Pythagoras himself; see my online notes for History of Mathematics [MATH 3040] on Section 3.4. Pythagorean Theorem and Pythagorean Triples).

**Definition.** A right triangle whose sides have integer lengths is a *Pythagorean triangle*. The lengths of the sides a Pythagorean triangle form a *Pythagorean triple*.

**Note.** To find all Pythagorean triangles (or equivalently, all Pythagorean triples) is to find all positive integer solutions of the equation $x^2 + y^2 = z^2$. We will find all Pythagorean triples in Theorem 16.1.

**Note 16.A.** We may assume that $x$ and $y$ are relatively prime, for if the greatest common divisor $(x, y) = d$, then we must have $d \mid z$, and so $(x/d)^2 + (y/d)^2 = (z/d)^2$, where $(x/d, y/d) = 1$ by Theorem 1.1. So if we can solve $x^2 + y^2 = z^2$ for $(x, y) = 1$, then we can find all solutions of $x^2 + y^2 = z^2$.

**Definition.** A solution $x = a$, $y = b$, $z = c$ of $x^2 + y^2 = z^2$ in which $a$, $b$, and $c$ are positive and the greatest common divisor $(a, b) = 1$ is called a *fundamental solution* of the equation. The corresponding triple, $(a, b, c)$, is called a *primitive Pythagorean triple.*

**Note.** It is easily shown that if $(x, y) = 1$ in $x^2 + y^2 = z^2$ then $(y, z) = (x, z) = 1$; this is Exercise 1 on page 129. So we see that in a fundamental solution $x = a$, $y = b$, $z = c$ we have no two of $, b, c$ have a common prime factor.

**Lemma 16.1.** If $a, b, c$ is a fundamental solution of $x^2 + y^2 = z^2$, then exactly one of $a$ and $b$ is even.

**Note.** By Lemma 16.1, $a^2 + b^2$ is odd in a fundamental solution of $x^2 + y^2 = z^2$, so $c^2$ must be odd. This is formalized in the next corollary.

**Corollary 16.A.** If $a, b, c$ is a fundamental solution, then $c$ is odd.

**Note.** We need some lemmas before classifying fundamental solutions to $x^2 + y^2 = z^2$.

**Lemma 16.2.** If $r^2 = st$ and $(s, t) = 1$, then both $s$ and $t$ are squares.

**Lemma 16.3.** Suppose that $a, b, c$ is a fundamental solution of $x^2 + y^2 = z^2$, and suppose that $a$ is even. Then there are positive integers $m$ and $n$ with $m > n$, $(m, n) = 1$, and $m \not\equiv n \pmod 2$ such that $a = 2mn$, $b = m^2 - n^2$, and $c = m^2 + n^2$.

**Lemma 16.4.** If $a = 2mn$, $b = m^2 - n^2$, and $c = m^2 + n^2$, then $a, b, c$ is a solution of $x^2 + y^2 = z^2$. If in addition, $m > n$, $m$ and $n$ are positive, $(m, n) = 1$, and $m \not\equiv n \pmod 2$, then $a, b, c$ is a fundamental solution.

**Note.** We can now combine Lemmas 16.3 and 16.4 into the following.

**Theorem 16.1.** All solutions $x = a$, $y = b$, $z = c$ to $x^2 + y^2 = z^2$, where $a, b, c$ are positive and have no common factor and $a$ is even, are given by $a = 2mn$, $b = m^2 - n^2$, $c = m^2 + n^2$, where $m$ and $n$ are any relatively prime integers, not both odd, and $m > n$.

**Note.** In the supplement to this section, The Group of Primitive Pythagorean Triples, we use Theorem 16.1 to put a group structure on all of the fundamental solutions to $x^2 + y^2 = z^2$ (that is, on all Pythagorean triples).

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