Section 21. Bounds for $\pi(x)$

Note. In this section we consider the number of primes in the set $\{1, 2, \ldots, n\}$ as a function of n. For x a real number, let $\pi(x)$ denote the number of primes less than or equal to x. We put an upper and lower bound on the function $\pi(x)/\log x$ and discuss this in the context of the Prime Number Theorem.

Note. A few values of function π are:

x	1	10	100	1000	10000	10^5	10^6	10^7
$\pi(x)$	0	4	25	168	1229	9592	78498	664579

We see that $\pi(x)$ is increasing, but at a slower rate than x is increasing. A somewhat detailed history of these ideas is given in a supplement to this section, so we give a brief history here. Both Adrien-Marie Legendre (September 18, 1752–January 10, 1833) and Carl Friedrich Gauss (April 30, 1777–February 23, 1855) conjectured that for large x, $\pi(x)$ is approximately equal to $x/\log x$. We quickly comment here that in these notes we use "$\log x$" to indicate the natural logarithm function, whereas Dudley uses "$\ln x$." The conjecture is known today as the Prime Number Theorem.

Note. Prime Number Theorem. As x increases without bound, the ratio of $\pi(x)$ to $x/\log x$ approaches 1.

Note. Pafnuty Chebyshev (May 16, 1821–December 8, 1894) in the 1840s computed constants c_1 and c_2, both close to 1, such that

$$c_1 < \frac{\pi(x)}{x/\log x} < c_2$$

(see Supplement. The Prime Number Theorem—History for more details and the values of c_1 and c_2). Bernhard Riemann (September 17, 1826–July 20, 1866) in a 9-page article "On the Number of Primes Less Than a Given Magnitude" (published in the November 1859 issue of *Monatsberichte der Königlich Preußischen Akademie der Wissenschaften zu Berlin*) formally introduced the zeta function and set the stage for the proof of the Prime Number Theorem (and laid the foundations of research that continues today). A translation appears in the appendix of Harold Edwards' *Riemann's Zeta Function*, Academic Press 1974 (reprinted by Dover Publications in 2001), and a translation is online on the Claymath.org website (accessed 3/6/2022). In 1896, Jacques Hadamard (December 8, 1865–October 17, 1963) and Charles de la Vallée Poussin (August 14, 1866–March 2, 1962) independently proved the Prime Number Theorem, using complex analysis and properties of Riemann's zeta function (in particular, by showing certain complex regions do not contain zeros of the zeta function).

Note. In this section, we prove a result similar to Chebyshev's, but much weaker. We will show

$$0.173 \approx \frac{1}{4}\log 2 < \frac{\pi(x)}{x/\log x} < 32\log 2 \approx 22.181 \text{ for } x \geq 2.$$

In the process, we establish some lemmas that we will use in the supplement to this section.

Note. Recall that for integer $n \geq 1$, we define n *factorial* as $n! = n\cdot(n-1)\cdot\cdots\cdot 3\cdot 2\cdot 1$, and for $n = 0$ we define $0! = 1$. The *binomial coefficient* is

$$\binom{n}{r} = \frac{n(n-1)(n-2)\cdots(n-r+1)}{r(r-1)\cdots(2)(1)} = \frac{n!}{r!(n-r)!},$$

where $n \geq r \geq 1$.

Note. The proof of the theorem in this section involves properties of $\binom{2n}{n}$. Table 1 (next page) gives the factorization for $n = 1, 2, \ldots, 20$. The table suggests some patterns:

1. All the primes between n and $2n$ appear with exponent 1.

2. None of the primes between $2n/3$ and n appear at all.

3. $\binom{2n}{n}$ is always divisible by $n + 1$.

4. Each prime-power is less than $2n$.

Notice that $\binom{2n}{n} = (2n)!/(n!)^2$. We are interested in the prime-power decomposition of $\binom{2n}{n}$. Recall that $[x]$ denotes the greatest integer less than or equal to x.

Lemma 21.1. The highest power of p that divides $n!$ is $[n/p]+[n/p^2]+[n/p^3]+\cdots$.

Note. We can use Lemma 21.1 to determine how many zeros there are at the end of 1984!. We do so by finding the highest power of 2 and the highest power of 5 in 1984!. For $p = 2$, the highest power dividing 1984! is

Table 1. Exponent of p in the prime-power decomposition of $\binom{2n}{n}$.

p \ n	2	3	5	7	11	13	17	19	23	29	31	37
1	1											
2	1	1										
3	2	0	1									
4	1	0	1	1								
5	2	2	0	1								
6	3	1	0	1	1							
7	3	1	1	0	1	1						
8	1	2	1	0	1	1						
9	1	0	1	0	1	1	1					
10	2	0	0	0	1	1	1	1				
11	2	1	0	1	0	1	1	1				
12	1	0	0	1	0	1	1	1	1			
13	2	0	2	1	0	0	1	1	1			
14	2	3	2	0	0	0	1	1	1			
15	3	2	1	0	0	0	1	1	1	1		
16	0	2	1	0	0	0	1	1	1	1	1	
17	1	3	1	0	1	0	0	1	1	1	1	
18	1	1	2	1	1	0	0	1	1	1	1	
19	2	1	2	1	1	0	0	0	1	1	1	1
20	1	2	1	1	1	1	0	0	1	1	1	1

$$\left[\frac{1984}{2}\right]+\left[\frac{1984}{4}\right]+\left[\frac{1984}{8}\right]+\left[\frac{1984}{16}\right]+\left[\frac{1984}{32}\right]+\left[\frac{1984}{64}\right]+\left[\frac{1984}{128}\right]+\left[\frac{1984}{256}\right]+\left[\frac{1984}{512}\right]$$
$$+\left[\frac{1984}{1024}\right]=992+496+248+124+62+31+15+7+3+1=1979.$$

For $p = 5$, the highest power dividing 1984! is

$$\left[\frac{1984}{5}\right]+\left[\frac{1984}{25}\right]+\left[\frac{1984}{125}\right]+\left[\frac{1984}{625}\right]=396+79+15+3=493.$$

So 1984! is divisible by $2 \times 5 = 10$ to the 493 power; that is, 1984! ends with 493 zeros.

Note. The next lemma is called "Legendre's Theorem" in Martin Aigner and Günter Zielger's *Proofs from THE BOOK*, 6th Edition, Springer (2018); see page 10.

Lemma 21.2 The highest power of p that divides $\binom{2n}{n}$ is

$$[2n/p]-2[n/p]+2[n/p^2]-2[n/p^2]+[2n/p^3]-2[n/p^3]+\cdots.$$

Lemma 21.3. For any x, $[2x]-x[x]\le 1$.

Note. The next lemma is a vital step in establishing our bounds on $\pi(x)/(x/\log x)$.

Lemma 21.4. Each prime-power in the prime-power decomposition of $\binom{2n}{n}$ is less than or equal to $2n$.

Note. Next we establish bounds on $\binom{2n}{n}$.

Lemma 21.5. For $n \geq 1$, we have $2^n \leq \binom{2n}{n} \leq 2^{2n}$.

Note. Three more lemmas, and we will be ready to prove our bounds on $\dfrac{\pi(x)}{x/\log x}$.

Lemma 21.6. For $n \geq 2$, we have $\pi(2n) - \pi(n) \leq (2n \log 2)/\log n$.

Lemma 21.7. For $n \geq 2$, we have $\pi(2n) \geq (n \log 2)/\log(2n)$.

Lemma 21.8. For $r \geq 1$, we have $\pi(2^{2r}) < 2^{2r+2}/r$.

Note. We now have the equipment to prove our main result.

Theorem 21.1. For $x \geq 2$, we have

$$\frac{1}{4}\log 2(x/\log x) \leq \pi(x) \leq (32 \log 2)(x/\log x).$$

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