Section 7. The Divisors of an Integer

Note. In this section, we consider two functions. Function d counts the number of divisors of a positive integer, and function σ sums the divisors of a positive integer. We give some properties of the functions and show how to use a prime-power decomposition of a positive integer to efficiently evaluate the functions. Some of this material is also covered in my online notes for Mathematical Reasoning on Section 6.9. Perfect Numbers, Mersenne Primes, Arithmetic Functions.

Note. Let n be a positive integer. Let $d(n)$ denote the *number of positive divisors of* n (including 1 and n). Let $\sigma(n)$ denote the *sum of the positive divisors of* n. Symbolically,

$$d(n) = \sum_{d \,|\, n} 1 \text{ and } \sigma(n) = \sum_{d \,|\, n} d.$$

Note 7.A. For example, with $n = 28$ we have the divisors 1, 2, 4, 7, 14, 28 and so

$$d(24) = \sum_{d \,|\, 28} 1 = 6 \text{ and } \sigma(n) = \sum_{d \,|\, n} d = 1 + 2 + 4 + 7 + 14 + 28 = 56.$$

If p is prime, then the only divisors are 1 and p so that $d(p) = 2$ and $\sigma(p) = p + 1$. In fact, we have $d(n) = 2$ and $\sigma(n) = n + 1$ only when n is prime (since 1 and n are divisors of n for every positive n). The only divisors of p^2 are 1, p, and p^2 so that $d(p^2) = 3$ and $\sigma(p^2) = 1 + p + p^2$. In Exercise 7.2 it is to be shown that $d(p^n) = n+1$ (and we can show that $\sigma(p^n) = (p^{n+1} - 1)/(p - 1)$, as is to be shown in Exercise 7.8). This observation, combined with the next result, allows us to count the number of divisors and any positive integer.

Theorem 7.1. If $p_1^{e_1} p_2^{e_2} \cdots p_k^{e_k}$ is the prime-power decomposition of n, then $d(n) = d(p_1^{e_1})d(p_2^{e_2}) \cdots d(p_k^{e_k})$.

Note. From Exercise 7.2 we have $d(p^n) = n + 1$, so we can use Theorem 7.1 to get an explicit formula for $d(n)$ based on the prime-power decomposition of n.

Corollary 7.A. If $p_1^{e_1} p_2^{e_2} \cdots p_k^{e_k}$ is the prime-power decomposition of n, then $d(n) = \prod_{i=1}^{k}(e_i + 1)$.

Exercise 7.4. Calculate $d(240)$.

Solution. We apply Theorem 7.1. To do so, we need the prime-power decomposition of 240. We have $240 = 2^4 \cdot 3 \cdot 5$. Now $d(2^4) = 5$ by Exercise 7.2. Since 3 and 5 are prime, $d(3) = d(5) = 2$ (by Exercise 7.2, say). So by Theorem 7.1 $d(240) = d(2^4 \cdot 3 \cdot 5) = d(2^4)d(3)d(5) = 5 \cdot 2 \cdot 2 = 20$.

Example 7.7. Show that $\sigma(2^n) = 2^{n+1} - 1$.

Solution. The divisors of 2^n are $1, 2, 2^2, 2^3, \ldots, 2^n$, so $\sigma(2^n) = 1+2+2^2+2^3+\cdots+2^n$. Notice that

$$1 + 2 + 2^2 + 2^3 + \cdots + 2^n = (2 - 1)(1 + 2 + 2^2 + 2^3 + \cdots + 2^n)$$

$$= (2 + 2^2 + 2^3 + \cdots + 2^n + 2^{n+1}) - 1 - 2 - 2^3 - \cdots - 2^n = 2^{n+1} - 1,$$

so $\sigma(2^n) = 2^{n+1} - 1$, as claimed. □

Exercise 7.8. Find $\sigma(p^n)$ for $n \in \mathbb{N}$.

Solution. The divisors of p^n are 1, p, p^2, $\ldots$, p^n, so that $\sigma(p^n) = 1+p+p^2+\cdots+p^n$. Notice that

$$(1+p+p^2+\cdots+p^n)(p-1) = (p+p^2+\cdots+p^n+p^{n+1})-1-p-p^2-\cdots-p^n = pn+1-1,$$

so that $\boxed{\sigma(p^n) = (p^{n+1}-1)/(p-1)}$. □

Note. We now turn our attention to a general formula for $\sigma(n)$.

Theorem 7.2. If $p_1^{e_1}p_2^{e_2}\cdots p_k^{e_k}$ is the prime-power decomposition of n, then $\sigma(n) = \sigma(p_1^{e_1})\sigma(p_2^{e_2})\cdots\sigma(p_k^{e_k})$.

Exercise 7.9. Calculate $\sigma(240)$ using Theorem 7.2.

Solution. We have the prime-power decomposition of $240 = 2^4 \cdot 3 \cdot 5$. Now $\sigma(3) = 4$ and $\sigma(5) = 6$ by Note 7.A. By Exercise 7.8, $\sigma(2^4) = ((2)^5-1)/((2)-1) = 31$. So by Theorem 7.2, $\sigma(240) = \sigma(2^4)\sigma(3)\sigma(5) = (31)(4)(6) = \boxed{744}$. □

Definition. A function f defined on the positive integers is *multiplicative* if

$$(m,n) = 1 \text{ implies } f(mn) = f(m)f(n).$$

Note. The advantage of multiplicative functions is that they can de evaluated by determining their values on powers of primes (s shown in Theorem 7.5 below). The two functions d and σ are multiplicative, as we show in the next two theorems.

Theorem 7.3. d is multiplicative.

Theorem 7.4. σ is multiplicative.

Theorem 7.5. If f is a multiplicative function and the prime-power decomposition of n is $p_1^{e_1} p_2^{e_2} \cdots p_k^{e_k}$, then $f(p_1^{e_1}) f(p_2^{e_2}) \cdots f(p_k^{e_k})$.

Note. In Section 9. Euler's Theorem and Function, we introduce a function ("Euler's function") and use the fact that it is multiplicative and Theorem 7.5 to compute its values.

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