Section 8. Perfect Numbers

Note. In this section, we define perfect numbers, discuss their history, and state some unsolved conjectures concerning them. Some of this material is also covered in my online notes for Mathematical Reasoning on Section 6.9. Perfect Numbers, Mersenne Primes, Arithmetic Functions.

Definition. A positive integer is *perfect* if it is equal to the sum of is positive divisors. That is, n is a perfect number if $\sigma(n) = 2n$.

Note. We have that 6 is a perfect number since its divisors are 1, 2, 3, and 6 and $1 + 2 + 3 + 6 = 2 \cdot 6$. Also, 28 is perfect since its divisors are 1, 2, 4, 7, 14, and 28 and $1 + 2 + 4 + 7 + 14 + 28 = 2 \cdot 28$. We can also show that 496 and 8,128 are perfect numbers.

Note. We now consider some history of perfect numbers (this same history is given in the Mathematical Reasoning [MATH 3000] notes mentioned above). The MacTutor History of Mathematics Archive's page on "Perfect Numbers" (on which most of this history is base; accessed 3/1/2022) mentions that is is not known as to when perfect numbers were first studied, but suggests that the Egyptians may have been aware of this idea (the page cites C. M. Taisbak's "Perfect numbers: A mathematical pun? An analysis of the last theorem in the ninth book of Euclid's Elements," *Centaurus* **20**(4), 269–275 (1976)). The page also mentions that Pythagoras took a mystical interest in perfect numbers. The first recorded mathematical result on perfect numbers appears around 300 BCE in Euclid's *Elements* in Book IX as Proposition 36: "If as many numbers as we please beginning from a unit are set out continuously in double proportion until the sum of all becomes prime, and if the sum multiplied into the last makes some number, then the product is perfect." In modern terminology, this translates into the claim: "If for some $k > 1$ we have $2^k - 1$ prime, then $2^{k-1}(2^k - 1)$ is a perfect number." This is our Theorem 8.1 below. In fact, for $2^k - 1$ to be prime it is necessary that k itself is prime (see my online notes for Mathematical Reasoning on Section 6.9. Perfect Numbers, Mersenne Primes, Arithmetic Functions, Exercise 6.93). Around 100 CE Nichomachus of Gerasa (circa 60 CE–120 CE) in his *Introductio Arithmetica* (a foundational work in classical algebra) gives a classification of numbers based on the idea of perfect numbers. By adding up what was called the "aliquot parts" of a number (what we would call the divisors of the number, excluding the number itself), he classified numbers (i.e., positive integers) as *deficient* (when the sum of the aliquot parts is less than the number), *superabundant* (when the sum of the aliquot parts is greater than the number; Dudley uses the term "abundant" as opposed to superabundant; see page 61), and *perfect* (when the sum of the aliquot parts equals the number). This idea of some type of "balance" with perfect numbers has been taken up by some in the religious and mystical community (Nichomachus himself made some strange observations). Nichomachus made several claims about perfect numbers, but provided no proofs. Some of his claims are true, some are false, and some are still open problems. In particular, he claimed that there are infinitely many perfect numbers. This and his other claims are bold, given that there were only four perfect numbers known at the time: 6, 28, 496, and 8128.

Note. Islamic mathematician and astronomer Thabit ibn Qurra (836–901) in his *Treatise on Amicable Numbers* explored when numbers of the form $2^n p$ are perfect, where p is prime. Islamic mathematician Ibn al-Haytham (965–1039) gave a partial converse of Euclid's Proposition IX.36 in his *Treatise on Analysis and Synthesis*. Ismail ibn Ibrahim ibn Fallus (1194–1239) wrote a treatise in which he gave a table of ten numbers that he claimed were perfect; the first seven were correct, but the last three were not. The fifth perfect number (33,550,336) was rediscovered (the results of Fallus seem to be unknown at the time in central Europe) and included in a manuscript dated 1461. Another mansucript by the same author included both the fifth and sixth perfect numbers (the sixth perfect number is 8,589,869,056). The only thing known about the author of these manuscripts is that he lived in Florence. The German scholar Johan Regiomontanus (June 6, 1436–July 6, 1476) in 1461 included the fifth perfect number in a manuscript he wrote in 1461. In 1536, Hudalrichus Regius in his *Ultriusque Arithmetices* observed that $2^{11}-1 = 2047 = 23 \cdot 89$ so that $2^{p-1}(2^p-1)$ is not a perfect number. That is, Regius has found the first prime p such that $2^{p-1}(2^p - 1)$ is not perfect. he also showed that $2^{13} - 1 = 8191$ is prime so that (by Euclid IX.36) $2^{12}(2^{13} - 1) = 33{,}550{,}336$ is perfect (this is another "discovery" of the fifth perfect number).

Note. In 1603, Italian mathematician Pietro Cataldi (April 15, 1548–February 11, 1626) created a table of primes up to 750 and used it to find the sixth perfect number (again) and the seventh perfect number (namely, 137,438,691,328); he also made some false claims. In a letter to French monk and math enthusiast Marin Mersenne (September 8, 1588–September 1, 1648) in 1640, Fermat used his "Little

Theorem" (Corollary 6.53) to show two of Cataldi's claims were wrong (he factored two numbers which Cataldi had claimed were prime). Fermat's letter inspired Mersenne to further explore prime numbers and perfect numbers. He published *Cogitata Physica Mathematica* in 1644 in which he claimed $2^p - 1$ is prime for several values of prime p; these prime numbers then yield perfect numbers $2^{p-1}(2^p - 1)$ by Euclid IX.36. Primes of the form $2^p - 1$ are now known as *Mersenne primes.* In 1732 Euler was the next to give a new perfect number (the first in 125 years); he proved that $2^{30}(2^{31} - 1) = 2{,}305{,}843{,}008{,}139{,}952{,}128$ is the eighth perfect number. In two manuscripts that Euler wrote but did not publish, he proved the converse of Euclid's Proposition IX.39. That is, he proved that every even perfect number is of the form $2^{p-1}(2^p - 1)$ where p is prime and $2^p - 1$ is a Mersenne prime (our Theorem 8.2 below). Of course this does not give all even perfect numbers explicitly, since there are unanswered questions about Mersenne primes. Skipping ahead quite a bit, according to the Wikipedia page "List of Mersenne Primes and Perfect Numbers" (accessed 3/1/2022), there are 51 known Mersenne primes and perfect numbers. The largest known perfect number is $2^{p-1}(2^p - 1)$ where $p = 82{,}589{,}933$, computed in late 2018; it has almost 50 million digits (more details can be found on the Great Internet Mersenne Prime Search (GIMPS) website) and $2^{82{,}589{,}933} - 1$ presently (March 4, 2022) stands as the largest known prime number (it has almost 25 million digits).

Note. Some unsolved problems concerning perfect numbers and Mersenne primes include:

1. Are there infinitely many perfect numbers?

2. Are there infinitely many Mersenne primes?

3. Are there any odd perfect numbers?

According to Gerstein (see page 340), it is known that no odd perfect number exists that is less than 10^{300}. We now return to the presentation given in Gerstein.

Note. We now state prove our version of Euclid's result from his *Elements*, Book IX Proposition 36.

Theorem 8.1 (Euclid). If $2^k - 1$ is prime, then $2^{k-1}(2^k - 1)$ is perfect.

Note. Euclid's Proposition IX.36 motivates us to consider conditions under which $2^k - 1$ is prime. First, observe that if k is a composite number, say $k = ab$, then

$$2^k - 1 = 2^{ab} - 1 = (2^a - 1)(2^{a(b-1)} + 2^{a(b-2)} + \cdots + 2^a + 1),$$

as can be established by distribution. That is, if $k = ab$ is composite (where $a > 1$ and $b > 1$) then $2^k - 1$ is composite since it is divisible by $2^a - 1$. This leads us to explore numbers of the form $2^p - 1$, where p is prime, to see the these numbers are themselves prime. For the first few prime numbers, we have

p	2	3	5	7	11	13
$2^p - 1$	3	7	31	127	2047	8191

Each of these $2^p - 1$ is prime, except for $p = 11$ and $2^{11} - 1 = 2047$ (because $2047 = 23 \cdot 89$; this is described in the history given above in connection with the work of Regius in 1536).

Definition. A prime number of the form $2^p - 1$, where p is prime, is a *Mersenne Prime*.

Note. In fact, the only *even* perfect numbers are of the form $2^{p-1}(2^p - 1)$, as Euler showed in the next result. The result is called "Euler's Theorem" or, to remove ambiguity, (since Euler proved hundreds, if not thousands, of theorems) the "Euclid-Euler Theorem."

Theorem 8.2 (Euler). If n is an even perfect number, then $n = 2^{p-1}(2^p - 1)$ for some prime p, and $2^p - 1$ is also prime.

Note. Dudley gives two versions of the argument that parameter $s = 1$ in his proof of Theorem 8.2 (see page 59). Only the first argument is given in our proof (in the supplement to this section of notes). In my online notes for Mathematical Reasoning (MATH 3000), the same result is given in Section 6.9. Perfect Numbers, Mersenne Primes, Arithmetic Functions; see Theorem 6.94. The proof given in the supplement to those notes uses Dudley's second version of the argument.

Note. Another idea based on the σ function is that of amicable numbers. The idea is that the sum of the divisors of one number m (excluding m itself) equals the sum of the divisors of the other number n (excluding n itself). You can see some of the history here, in that a number was historically not considered as a divisor of itself. Notice that if we exclude the number n itself from the list of divisors of n, then n

is a perfect number if it is a sum of its (resulting "proper") divisors. Formally, for amicable numbers we have the following definition.

Definition. Positive integers m and n are *amicable* (or are an *amicable pair*) if $\sigma(m) = \sigma(n) = m + n$; that is, if $\sigma(m) - m = n$ and $\sigma(n) - n = m$.

Note. Since $220 = 2^2 \cdot 5 \cdot 11$ and $284 = 2^2 \cdot 71$, then

$$\sigma(220) - 220 = \sigma(2^2)\sigma(5)\sigma(11) - 200$$

$$= \frac{(2)^{(2)+1} - 1}{(2) - 1}((5) + 1)((11) + 1) - 220 = (7)(6)(12) - 220 = 504 - 220 = 284,$$

and

$$\sigma(284) - 284 = \sigma(2^2)\sigma(71) - 284 = \frac{(2)^{(2)+1} - 1}{(2) - 1}((71) + 1) - 284$$
$$= (7)(72) - 284 = 504 - 284 = 220.$$

One can also show (as is required in Exercise 8.1) that 1184 and 1210 are amicable numbers.

Note. According to the Wikipedia page on Amicable Numbers, the first 10 pairs of amicable numbers is

#	1	2	3	4	5	6	7	8	9	10
m	220	1,184	2,620	5,020	6,232	10,744	12,285	17,296	63,020	66,928
n	284	1,210	2,924	5,564	6,368	10,856	14,595	18,416	76,084	66,992

As of March 4, 2022, there are over 1,227,317,909 known amicable pairs according to the Amicable Pairs List website (accessed 3/4/2022). According to Dudley (see page 61): "There are as yet no general theorems on amicable numbers as beautiful as Euclid's and Euler's theorems on perfect numbers. Perhaps they remain to be discovered." There are many open problems concerning amicable pairs.

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