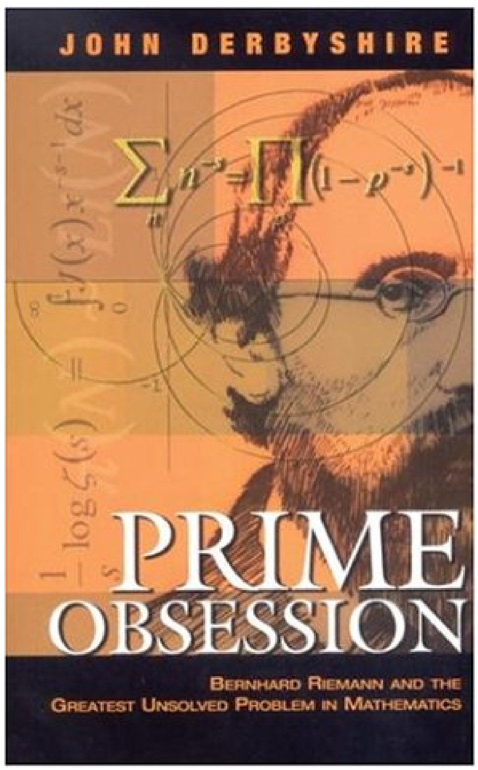
Supplement. The Prime Number Theorem—History

Note. In this supplement, we give a survey of the history of the Prime Number Theorem ("PNT"). Our main reference is John Derbyshire's *Prime Obsession: Bernhard Riemann and the Greatest Unsolved Problem in Mathematics*, Washington, DC: Joseph Henry Press (2003). As the title suggests, this book addresses the Riemann Hypothesis concerning the zeros of the Riemann zeta function, but the first part of the book is on the Prime Number Theorem.



The Prime Number Theorem concerns an asymptotic approximation of the number of primes less than a given integer. We start with reviewing some results concerning prime numbers from Elementary Number Theory (MATH 3120) which are of historical interest.

Note. In these notes, we restrict our attention to positive integers. Recall that an integer that is greater than 1 and has no positive integer divisors other that 1 and itself if a *prime number* and an integer greater than 1 that is not prime is a *composite number*. Some preliminary results on prime numbers are (1) every integer greater than 1 is divisible by a prime, and (2) every integer greater than 1 can be written as a product of primes (see my online notes for Elementary Number Theory on Section 2. Unique Factorization; notice Lemmas 2.1 and 2.2). The first historically significant result concerning primes numbers is Euclid's proof that there are infinitely many primes (for a proof, see Theorem 2.1 of the Elementary Number Theory notes just referenced). This appears in Euclid's *Elements of Geometry* as Proposition 20 in Book IX, where it is stated as "Prime numbers are more than any assigned multitude of prime numbers." Euclid's proof is online in David Joyce's online version of Euclid's *Elements* (accessed 3/30/2022). For a brief discussion of the number theory contained in Euclid's *Elements*, see my online notes for Introduction to Modern Geometry (MATH 4157/5157) on Section 2.4. Books VII and IX. Number Theory.

Note. A technique for finding prime numbers is the Seive of Eratosthenes, named for Eratosthenes of Cyrene (275 BCE–194 BCE). This technique involves eliminating composite numbers from a list of positive integers greater than 1 by first eliminating multiples of 2 greater than 2, then eliminating multiples of 3 greater than 3, then eliminating multiples of 5 greater than 5, etc. If all multiples of primes less than or equal to n have been removed, then all numbers less than n^2 which remain must be prime. Wikipedia has a nice animated GIF of the Sieve of Eratosthenes which finds all prime numbers between 2 and 120 by eliminating multiples of 2, 3, 5, 7, and 11 (accessed 3/30/2022). Some additional historical information on Eratosthenes is in my online notes Section 2. Unique Factorization.


Images of Euclid (circa 325 BCE–circa 265 BCE) from World History Encyclopedia and Eratosthenes (276 BCE–194 BCE) from MacTutor History of Mathematics Archive (accessed 3/30/2022).

Note. An extremely important result is the fact that every (positive) integer is "made up" of prime numbers. That is, every (positive) integer is a product of powers of prime numbers and this representation is unique. This is the Fundamental Theorem of Arithmetic (also sometimes called the Unique Factorization Theorem). In Elementary Number Theory (MATH 3120) it is stated as (see Theorem 2.2 in the aforementioned Section 2. Unique Factorization:

> **The Unique Factorization Theorem or The Fundamental Theorem of Arithmetic.**
>
> Any positive integer greater than 1 can be written as a product of primes in one and only one way.

The fact that every positive integer greater than 1 is a product of primes is in Euclid's *Elements* as Proposition 14 of Book IX, where it is stated as: "If a number be the least that is measured by prime numbers, it will not be measured by any other prime number except those originally measuring it." In modern terminology, this would be read: "a least common multiple of several prime numbers is not a multiple of any other prime number." However, this only partially addresses the uniqueness of the representation (see the Wikipedia page on the Fundamental Theorem of Arithmetic; accessed 3/30/2022).

Note. Recall that a positive integer is *perfect* is it is twice the sum of its (positive) divisors; see my online Elementary Number Theory notes on Section 8. Perfect Numbers. Another result in the *Elements* is Proposition 36 of Book IX which states (in modern terminology): "If for some $k > 1$ we have $2^k - 1$ prime, then $2^{k-1}(2^k - 1)$ is a perfect number." A proof is given in the online notes just mentioned. In fact, it is for $2^k - 1$ to be prime it is necessary that k itself is prime (see my online notes for Mathematical Reasoning [MATH 3000] on Section 6.9. Perfect Numbers, Mersenne Primes, Arithmetic Functions; see Exercise 6.93). Islamic mathematician Ibn al-Haytham (9651039) gave a partial converse of Euclids Proposition IX.36 in his *Treatise on Analysis and Synthesis*. Moving significantly forward in time, Pierre de Fermat (August 17, 1601–January 12, 1665) wrote a letter to to French monk and math enthusiast Marin Mersenne (September 8, 1588September 1, 1648) in 1640 concerning perfect numbers. Fermats letter inspired Mersenne to further explore prime numbers and perfect numbers. Mersenne published *Cogitata Physica Mathematica* in 1644 in which he claimed $2^p - 1$ is prime for several values of prime

p; these prime numbers then yield perfect numbers $2^{p-1}(2^p - 1)$ by Euclid IX.36. Primes of the form $2^p - 1$ are now known as *Mersenne primes.* Some conjectures on Mersenne primes are listed in Section 8. Perfect Numbers. In two manuscripts that Leonhard Euler wrote but did not publish, he proved the converse of Euclid's Proposition IX.39. That is, he proved that every even perfect number is of the form $2^{p-1}(2^p-1)$ where p is prime and 2^p-1 is a Mersenne prime (this is given as Theorem 8.2 in Section 8. Perfect Numbers). This history is base on the MacTutor History of Mathematics Archive's page on "Perfect Numbers" (accessed 3/30/2022).


Images of Pierre de Fermat (August 17, 1601-January 12, 1665) from Fermat's Library website and Marin Mersenne (September 8, 1588–September 1, 1648) from MacTutor History of Mathematics Archive (accessed 3/30/2022).

Note. We mention two more results on prime numbers, before focusing on the Prime Number Theorem. We are interested in finding specific primes or finding prime-containing intervals of real numbers. In 1837 Peter Lejeune Dirichlet (February 13, 1805-May 5, 1859) proved:

Dirichlet's Theorem. For positive integers a and b, where $(a, b) = 1$, there are infinitely many primes of the form $an+b$, where n is a positive integer. That is, there are infinitely many primes that are congruent to b modulo a.

Dirichlet's Theorem is stated as Theorem 22.B in Section 22. Formulas for Primes. A result concerning locations of primes was conjectured in 1845 by Joseph Bertrand (March 11, 1822-April 5, 1900) and verified up to three million. It was proved by Pafnuty Chebyshev (May 16, 1821-December 8, 1894). The result is:

Bertrand's Postulate/Bertrand's Theorem/Bertrand-Chebyshev Theorem/Chebyshev's Theorem.

For all integers $n \geq 2$, there is a prime p such that $n < p < 2n$.

Bertrand's Postulate is stated as Theorem 22.2 in Section 22. Formulas for Primes, where a proof is given.

Note. Leonhard Euler (April 15, 1707–September 18, 1783) in 1737 published his *Variae observationes circa series infinitas* ("Various Observations about Infinite Series") in which he introduced analysis techniques into the study of number theory for the first time. He defined a function (later called the zeta function by Riemann), that yields the sum of a p-series when $p > 1$:

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = 1 + \frac{1}{2s} + \frac{1}{3^s} + \frac{1}{4^s} + \cdots \text{ where } s > 1.$$

He establishes the relationship that

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \left(\frac{1}{1-2^{-s}}\right)\left(\frac{1}{1-3^{-s}}\right)\left(\frac{1}{1-5^{-s}}\right)\left(\frac{1}{1-7^{-s}}\right)\left(\frac{1}{1-11^{-s}}\right)\cdots$$

$$= \prod_{p \text{ prime}} \frac{1}{1-p^{-s}} \text{ for } s > 1.$$

(see Wikipedia's Proof of the Euler Product Formula for the Riemann Zeta Function; accessed 3/31/2022). Taking a limit as $s \to 1^+$ allows us to conclude (as did Euclid) that there are an infinite number of primes since this limit yields (with some due apologies for the informal analysis) $\sum_{n=1}^{\infty} \frac{1}{n} = \prod_{p \text{ prime}} \frac{1}{1-p}$, and we know that the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges, so we know that the product must diverge and hence must be a product over an infinite number of primes p. In addition, Euler solved the "Basel Problem" in 1735 by showing $\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$, which he published in his 1740 *De summis serierum reciprocarum*. In fact, he found values for all even s. You can view this original work online (in Latin) at the Euler Archive on *De summis serierum reciprocarum* (accessed 3/31/2022). A nice reference on Euler's results along these lines is Raymond Ayoub's "Euler and the Zeta Function," *The American Mathematical Monthly*, **81**, 1067–86 (December 1974); this can be found online on the MAA website (accessed 3/31/2022). A more general lecture on Euler is my 2007 online Leonard Euler—Happy 300th Birthday! presentation.

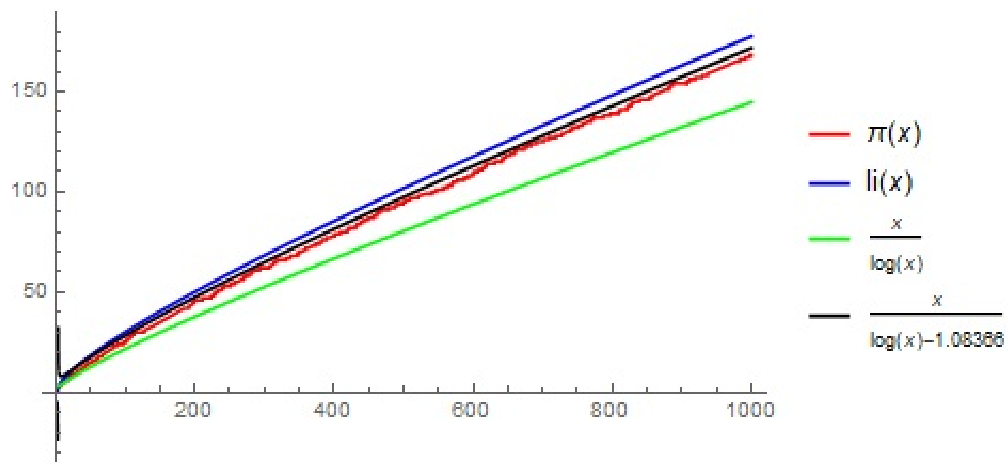
Image from the MacTutor History of Mathematics Archive page on Euler

Note. The function $\pi(x)$ is defined as the number of primes less than or equal to x; see my online notes for Elementary Number Theory (MATH 3120) on Section 21. Bounds for $\pi(x)$. It is sometimes called the *prime counting function* (see Derbyshire's *Prime Obsession*, page 38). The Prime Number Theorem claims that for large values of x, $\pi(x)$ is approximately $x/\ln x$. More precisely:

The Prime Number Theorem.

$$\lim_{x\to\infty} \frac{\pi(x)}{x/\log x} = 1.$$

You may also see this written as $\pi(x) \sim x/\log x$. A graph of $\pi(x)$ alongside a graph if $x/\log x$ is in the following image from an MAA website on "The Origin of the Prime Number Theorem: A Primary Source Project for Number Theory Students":



The image includes some other functions we will discuss. The first published result on a Prime Number Theorem type of result is due to Adrien-Marie Legendre (September 18, 1752–January 9, 1833). In his *Essai sur la Théorie des Nombres* [Essay on the Theory of Numbers] (1797–98), he conjectured that $\pi(x) \sim \dfrac{x}{A\log x + B}$ for some numbers A and B, "to be determined." He came to this idea by considering tables of prime numbers (some by Juri Vega, whose data is given on the MAA website mentioned above). In the second edition of his book (1808), he refined his estimate to $\pi(x) \sim \dfrac{x}{\log x - 1.08366}$ (notice that this function is also graphed in the image above).

An 1820 watercolor caricature of Legendre from the MacTutor History of Mathematics Archive page on Legendre; this is the only image of him.

Note. Though Legendre was the first to publish, it seems that Carl Friedrich Gauss (April 30, 1777–February 23, 1855) was the first to consider the question of the asymptotic behavior of $\pi(x)$. In 1792–93, he considered extensive tables of prime numbers (he spent some of his spare time creating such tables)and came to the conclusion that the density with which primes occur in a neighborhood of integer n is $1/\log n$, so that the number of primes in the interval $[a, b]$ is approximately $\displaystyle\int_a^b \frac{dx}{\log x}$ (see page 602 of J. L. Goldstein's "A History of the Prime Number Theorem," *The American Mathematical Monthly*, **80**(6), 599–615 (1973); this paper is available through JSTOR (accessed 3/31/2022). Gauss approximation for

$\pi(x)$ is then $\displaystyle\int_2^x \frac{dt}{\log t}$; this is called the *logarithmic integral* function and is denoted Li(x) (or sometimes li(x), as in the graph above). However, Gauss never publishes his results. The main source of material on his contributions are in a letter of December 24, 1849 which he wrote the the astronomer Johann Franz Encke (after whom a well-known short-term comet is named). In the letter, he explains his use of tables of primes (including the tables of Vega's mentioned above) and his approximation of $\pi(x)$ with the logarithmic integral Li(x). A translation of the letter is given in a appendix to Goldstein's "A History of the Prime Number Theorem." It was not unusual for Gauss to make claims such as this (for example, he claimed to know about hyperbolic geometry sometime between about 1790 and 1815 but said nothing about it until it was independently explained by Lobatschewsky and Bolyai around 1830; see my online presentation on Hyperbolic Geometry).

Image from the MacTutor History of Mathematics Archive page on Gauss

Note. We return to a discussion of Dirichlet's Theorem (Theorem 22.B of Section 22. Formulas for Primes). It is stated above in terms of there being infinitely many primes of the form $an + b$, where n is a positive integer and a and b are relatively prime. It is also common to describe it in terms of the arithmetic progression $b, b + a, b + 2a, b + 3a, \ldots$, for which Dirichlet's Theorem implies the sequence contains infinitely many primes. Inspired by earlier work of Euler (who we have already credited with introducing analytic techniques to the study of number theory, but Euler's arguments sometimes lacked rigor by modern standards), Dirichlet applied the analytic idea of L-series (which he also introduced). His result is described on Wikipedia's webpage on "Dirichlet's Theorem on Arithmetic Progressions" as: "The theorem represents the beginning of rigorous analytic number theory." A proof can be found in my online notes (in preparation) on Theory of Numbers (MATH 5070); see Theorem 10.5 of Chapter 10. Primes in an Arithmetic Progression.

Johann Lejeune Dirichlet (February 13, 1805–May 5, 1859)

Image from the MacTutor History of Mathematics Archive page on Dirichlet

Note. The next steps leading to the proof of the Prime Number Theorem are contained in two memoirs by Pafnuty Chebyshev (May 16, 1821–December 8, 1894) which were published in 1851 and 1852. Chebyshev defined the functions

$$\theta(x) = \sum_{p \le x} \log p \text{ and } \psi(x) = \sum_{p^m \le x} \log p,$$

where p runs over primes and m over positive integers. Chebyshev proved that the Prime Number Theorem is equivalent to either of the following two limits:

$$\lim_{x\to\infty} \frac{\theta(x)}{x} = 1 \text{ and } \lim_{x\to\infty} \frac{\psi(x)}{x} = 1,$$

and that *if* $\lim_{x\to\infty} \theta(x)/x$ exists then its value is 1. We mentioned Chebyshev's bound on $\dfrac{\pi(x)}{x/\log x}$ in Section 21. Bounds for $\pi(x)$. Specifically, he proved

$$0.92129 \le \liminf_{x\to\infty} \frac{\pi(x)}{x/\log x} \le 1 \le \limsup_{x\to\infty} \frac{\pi(x)}{x/\log x} \le 1.10555.$$

Image from the MacTutor History of Mathematics Archive page on Chebyshev

Chebyshev's results were insufficient to prove the Prime Number Theorem. His results were reprinted in 1899 in "Sur la fonction qui détermine la totalité de nombres premiers inférieurs à une limite don'ee," *Oeuvres*, **1**, 27–48 (1899), and "Memoire sur les nombres premiers," *Oeuvres*, **1**, 49–70 (1899); see page 606 of J. L. Goldstein's "A History of the Prime Number Theorem," mentioned above.

Note. Bernhard Riemann (September 17, 1826–July 20, 1866) in a 9-page article "On the Number of Primes Less Than a Given Magnitude" (published in the November 1859 issue of *Monatsberichte der Königlich Preußischen Akademie der Wissenschaften zu Berlin*) formally introduced the zeta function and set the stage for the proof of the Prime Number Theorem (and laid the foundations of research that continues today). A translation appears in the appendix of Harold Edwards' *Riemann's Zeta Function*, Academic Press 1974 (reprinted by Dover Publications in 2001), and a translation is online on the Claymath.org website (accessed 3/6/2022). Riemann's definition of $\zeta(s)$ for $\mathrm{Re}(z) > 1$ is the same as Euler's for $s > 1$:

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p \text{ prime}} \frac{1}{1 - p^{-s}} \text{ for } \mathrm{Re}(s) > 1.$$

Riemann extended $\zeta(s)$ to the rest complex plane, except $s = 1$. The zeta function is the meromorphic on $\mathbb{C}$ with a simple pole at $s = 1$ only. This is established in Complex Analysis 2 (MATH 5520) in VII.8. The Riemann Zeta Function (though the class traditionally does not reach this point). Riemann's zeta function has "trivial zeros" consisting of each negative even integer, $-2, -4, -6, \ldots$. Any other zeros of $\zeta(s)$ are called "nontrivial zeros." If the nontrivial zeros are located in real part less than one, then the Prime Number Theorem will follow; see page 156 of John Derbyshire's *Prime Obsession: Bernhard Riemann and the Greatest Unsolved Problem in Mathematics*, Washington, DC: Joseph Henry Press (2003). Riemann did not show that the nontrivial zeros have real part less than one, and so he was unable to prove the Prime Number Theorem. As discussed below, the Prime Number Theorem *was* proved in 1896, in fact, by showing the nontrivial zeros of the zeta function have real part less than one. But knowing that $\pi(x) \sim x/\log x$ leads to the next question of "How good is the approximation?," or equivalently "How bad is the error term when we approximate $\pi(x)$ by $x/\log x$?" "Riemann's 1859 paper gave an exact expression for the error term. That expression…involves all the non-trivial zeros of the zeta function, so the key to understanding the error term is hidden in among the zeros somehow." This quote is from page 234 of Derbyshire. So even though the Prime Number Theorem has been proved (as we consider next), it is still of interest to locate the nontrivial zeros of $\zeta(s)$. Riemann speculates in his 1859 paper that the nontrivial zeros all have real part equal to $1/2$ (and so lie on vertical line $\text{Re}(z) = 1/2$ in the complex plane). This is known as the Riemann Hypothesis and is probably the most famous unsolved mathematical conjecture today. If Riemann's Hypothesis is true, then the error term is of size $x^{1/2} \log x$ and this gives "the perfect balance of the zeros, and of the prime" (according to Brian Conrey's survey *Riemann's Hypothesis*; accessed 4/6/2022).

Image from the MacTutor History of Mathematics Archive page on Riemann

Note. Jacques Hadamard (December 8, 1865–October 17, 1863) developed in 1893 a theory of (complex) entire functions of finite order. Hans von Mangoldt (May 18, 1854–October 27, 1925) in "Auszug aus einer Arbeit unter dem Titel: Zu Riemann's Abhandlung uber die Anzahl der Primzahlen unter einer gegebenen Grösse," [Excerpt from a Work Entitled: On Riemann's Treatise on the Number of Prime Numbers below a Given Size] *Sitz Konig. Preus. Akad. Wiss.* zu Berlin, 337–350, 883–895 (1894) gave rigorous proofs of two claims made in Riemann's 1859 paper using Hadamard's results. Working independently, Hadamard and Charles-Jean de la Vallée Poussin (August 14, 1866–March 2, 1962) showed that $\zeta(s)$ has no zeros on $\mathrm{Re}(z) = 1$, from which the Prime Number Theorem follows. Some of this history is from Tom Apostol's "A Centennial History of the Prime Number Theorem," available on the CalTech website (accessed 4/6/2022). The references for Hadamard and Poussin's work are:

1. Jacques Hadamard, "Sur la distribution des zeros de la fonction $\zeta(s)$ et ses consequences arithmetiques," *Bull. Soc. Math. de France*, **24**, 199–220 (1896).

2. Charles de la Vallée Poussin, "Recherches analytiques sur la theorie des nombres premiers. Premiere partie. La fonction $\zeta(s)$ de Riemann et les nombres premiers en general" [Analytical Research on the Theory of Prime Numbers. First Part. The Riemann Function $\zeta(s)$ and Prime Numbers in General], *Ann. Soc. Sci. Bruxelles*, **20**, 183–256 (1896).

These proofs heavily depend on the theory of functions of a complex variable and this work falls under the category of analytic number theory.

Jacques Hadamard Charles de la Vallée Poussin

Images from the MacTutor History of Mathematics Archive biographies on Hadamard and Poussin

Note. We mention one more analytic proof of the Prime Number Theorem. In 1980, Donald J. Newman (July 27, 1930–March 28, 2007) gave a (relatively) simple proof of the Prime Number Theorem which uses the standard results from complex analysis of Cauchy's Integral Formula and Cauchy's Integral Theorem; for statements and proofs of these, see my online notes for Complex Variables (MATH 4337/5337) on Section 4.46. Cauchy-Goursat Theorem (Theorem 4.46.A is a statement of Cuachy's Theorem) and Section 4.50. Cauchy Integral Formula, or my motes for Complex Analysis 1 [MATH 5510] on Section IV.5. Cauchy's Theorem and Integral Formula. Newman's paper is "Simple Analytic Proof of the Prime Number Theorem," *The American Mathematical Monthly*, **87**(9), 693–696 (1980). A copy is online on the SUNY Stonybrook website. A similar proof with more details is given in D. Zagier, "Newman's Short Proof of the Prime Number Theorem," *The American Mathematical Monthly*, **104**(8), 705–708 (1997); this paper has the cute subtitle "Dedicated to the Prime Number Theorem on the occasion of its 100th birthday." A copy is online on Zagier's webpage. An expository paper (expository, but stilled crammed full of mathematical equations) further discussed Newman's proof in J. Korevaar's "On Newman's Quick Way to the Prime Number Theorem," *Mathematical Intelligencer*, **4**(3), 109–115 (1982). A copy is online on Korevaar's webpage. Each of these papers were accessed 4/6/2022.

Donald J. Newman (image from Korevaar's paper, page 108.)

Note. Following the 1896 analytic proofs of the Prime Number Theorem by Hadamard and de la Vallée Poussin, some started to look for a proof that did not depend on complex analysis. Such a proof is called an "elementary proof," though the term "elementary" is not to mean "simple," but instead refers to the fact that such a proof would only depend on traditional (non-analytic) number theoretic techniques. There was early skepticism that such a proof would be found. Godfrey Harold Hardy (February 7, 1877–December 1, 1947) famously commented in 1921:

> "No elementary proof of the prime number theorem is known, and one may ask whether it is reasonable to expect one. ... A proof of such a theorem, not fundamentally dependent on the theory of functions, seems to me extraordinarily unlikely. ... If anyone produces an elementary proof of the prime number theorem, he will show that these views are wrong, that the subject does not hang together in the way we have supposed, and that it is time for the books to be cast aside and for the theory to be rewritten."

So it was somewhat of a surprise when, in 1948, Atle Selberg (June 14, 1917–August 6, 2007) and Paul Erdős (Mach 26, 1913–September 20, 1996) found an elementary proof! Both proofs depend on an inequality due to Selberg. Selberg and Erdős were both at Princeton at this time and had discussed the Prime Number Theorem. Unfortunately, Selberg and Erdős could not agree on how to publish the result, and each individually published a version of the proof. Their work appeared in 1949 as:

1. Atle Selberg, "An Elementary Proof of the Prime-Number Theorem," *Annals of Mathematics*, **50**(2), 305–313 (1949).

2. Paul Erdős, "On a New Method in Elementary Number Theory which leads to an Elementary Proof of the Prime Number Theorem," *Proceedings of the National Academy of Sciences*, **35**, 374–384.

The history of the Selberg/Erdős controversy is detailed in D. Goldfield's "The Elementary Proof of the Prime Number Theorem: An Historical Perspective" in *Number Theory: New York Seminar 2003*, eds. D. Chudnovsky, G. Chudnovsky, and N. Nathanson, 179–192 (Spring, 2004) (this contains the Hardy quote given above). This is available online on Goldfeld's webpage. Another perspective on the controversy (including the account of first-hand witnesses) is given in Joel Spencer and Ronald Graham's "The Elementary proof of the Prime Number Theorem," *The Mathematical Intelligencer*, **31**(3), 18–23 (2009). This is available online on Ron Graham's webpage. Another interesting document is Ashvin Swaminathan's "On the Selberg-Erdős Proof of the Prime Number Theorem," apparently the result of a sophomore project for a class he took at Harvard! This gives a presentation of the elementary proof, including Selberg's symmetry formula, Mangoldt's function, and Möbius inversion. A copy is online on Swaminathan's webpage. Each of these papers were accessed 4/6/2022.

Atle Selberg Paul Erdős

Images from the MacTutor History of Mathematics Archive biographies on Selberg and Erdős

Note. We conclude this supplement by mentioning Graham J. O. Jameson's book *The Prime Number Theorem*, London Mathematical Society Student Texts, Series Number 53, Cambridge University Press (2003). The Prime Number Theorem is covered in the first three chapters. Chapter 3 gives two analytic proofs of the Prime Number Theorem. Chapter 6, "An 'Elementary' Proof of the Prime Number Theorem," covers the proof of by Selberg and Erdős. I have online notes in preparation based on this book for use as a supplement to either Elementary Number Theory (MATH 3120) or Theory of Numbers (MATH 5070): Prime Number Theorem Class Notes.

Revised: 4/6/2022