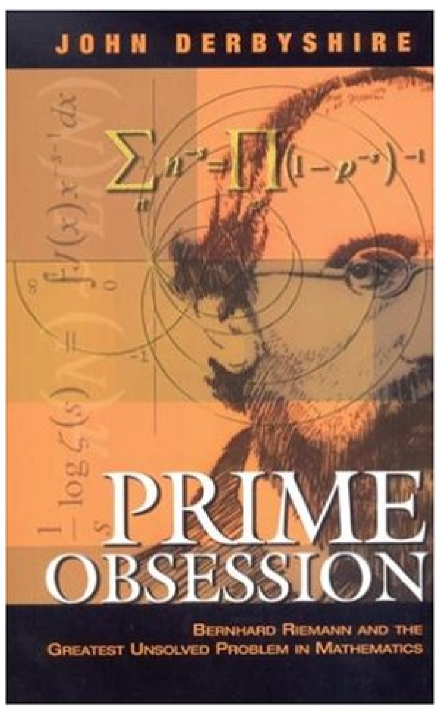
Supplement. The Riemann Hypothesis-History

Note. This supplement is a continuation of my online supplement on Supplement. The Prime Number Theorem—History. Both of these are meant to be used as supplements to the junior-level class Elementary Number Theory (MATH 3120) and the graduate-level class Number Theory (MATH 5070).

Note. As with the supplement on the history of the Prime Number Theorem, our main reference is John Derbyshire's *Prime Obsession: Bernhard Riemann and the Greatest Unsolved Problem in Mathematics*, Washington, DC: Joseph Henry Press (2003).



We start with a brief review of the Prime Number Theorem and how it relates to the Riemann zeta function.

Note. Let $\pi(x)$ denote the number of primes less than or equal to x. We saw in Supplement. The Prime Number Theorem—History that Adrien-Marie Legendre (September 18, 1752–January 9, 1833) and Carl Friedrich Gauss (April 30, 1777–February 23, 1855) were the first to address the asymptotic behavior of $\pi(x)$. Both based their approximations on the study of tables of integrals. Gauss proposed (though he never published it) that the logarithmic integral $\mathrm{Li}(x) = \int_2^x \frac{dt}{\log t}$. In the early 1850s, Pafnuty Chebyshev (May 16, 1821–December 8, 1894) put bounds on $\frac{\pi(x)}{x/\log x}$, strongly suggesting that $\pi(x)$ is closely approximated by $x/\log x$.

Note. Bernhard Riemann (September 17, 1826–July 20, 1866) in a 9-page article "On the Number of Primes Less Than a Given Magnitude" (published in the November 1859 issue of *Monatsberichte der Königlich Preußischen Akademie der Wissenschaften zu Berlin*) formally introduced the zeta function and set the stage for the proof of the Prime Number Theorem. A translation appears in the appendix of Harold Edwards' *Riemann's Zeta Function*, Academic Press 1974 (reprinted by Dover Publications in 2001), and a translation is online on the Claymath.org website (accessed 3/6/2022). Riemann's definition of $\zeta(s)$ for $\mathrm{Re}(s) > 1$:

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p \text{ prime}} \frac{1}{1-p^{-s}} \text{ for } \mathrm{Re}(s) > 1,$$

which agrees with an earlier observation by Leonhard Euler (April 15, 1707–September 18, 1783) in 1737 which considered the case s real and $s > 1$. Riemann extended $\zeta(s)$ to the rest complex plane, except $s = 1$. The zeta function is the meromorphic on $\mathbb{C}$ with a simple pole at $s = 1$ only. This is established in Complex Analysis 2 (MATH 5520) in Section VII.8. The Riemann Zeta Function (though the class traditionally does not reach this point). We will appeal to some of the results from the Complex Analysis 2 class in this supplement.

Image from the MacTutor History of Mathematics Archive page on Riemann

Note. The Prime Number Theorem claims that $\pi(x)$ is asymptotically approximated by $x/\log x$ (or by $\text{Li}(x)$). This is denoted $\pi(x) \sim x/\log x$ and $\pi(x) \sim \text{Li}(x)$. In fact, $x/\log x \sim \text{Li}(x)$, so that these two approximations are equivalent. See my online notes for The Prime Number Theorem on Section 1.1. Counting Prime Numbers. In 1896, Jacques Hadamard (December 8, 1865–October 17, 1963) and Charles de la Vallée Poussin (August 14, 1866–March 2, 1962) independently proved the Prime Number Theorem. Their proofs involved the location of certain zeros of the zeta function (namely, they both showed that the "'nontrivial zeros" of the zeta function lie in $\text{Re}(z) < 1$). It may be surprising that a result in number theory is based so heavily on results from analysis. A search started for a proof that did not depend on analysis (commonly called an "elementary proof"). Such a proof was found in 1949 by Alte Selberg (June 14, 1917–August 6, 2007) and Paul Erdős (March 26, 1913–September 20, 1996).

Note. In ETSU's Complex Analysis graduate sequence (MATH 5510/5520), the Riemann zeta function $\zeta(z)$ is defined in three steps by defining it over different regions of the complex plane; see Section VII.8. The Riemann Zeta Function. The extension is not accomplished by "analytic continuation," but by relating the zeta function to the gamma function. First, as Riemann himself did (and Euler when z is real), for $\text{Re}(z) > 1$ we start with:

$$\zeta(z) = \sum_{n=1}^{\infty} \frac{1}{n^z}.$$

Next, as the first step, the definition is extended to $\text{Re}(z) > 0$ by giving a definition in terms of the gamma function $\Gamma(z)$ (see my online Complex Analysis notes on Section VII.7. The Gamma Function):

$$\zeta(z) = \frac{1}{\Gamma(z)} \left(\int_0^1 \left(\frac{1}{e^t - 1} - \frac{1}{t} \right) t^{z-1}\, dt + (z-1)^{-1} + \int_1^{\infty} \frac{t^{z-1}}{e^t - 1} \right).$$

Of course, it must be shown that this definition agrees with the definition on $\text{Re}(z) > 1$. For the second step (after much motivation), for $-1 < \text{Re}(z) < 1$ $\zeta(z)$ is defined as

$$\zeta(z) = \frac{1}{\Gamma(z)} \left(\int_0^1 \left(\frac{1}{e^t - 1} - \frac{1}{t} + \frac{1}{2} \right) t^{z-1}\, dt - \frac{1}{2z} + \int_1^{\infty} \left(\frac{1}{e^1 - 1} - \frac{1}{t} \right) t^{z-1}\, dt \right).$$

For the third step, for $\text{Re}(z) < 0$ $\zeta(z)$ is defined as:

$$\zeta(z) = 2(2\pi)^{z-1}\Gamma(1-z)\zeta(1-z)\sin\left(\frac{1}{2}\pi z\right).$$

This development leads to the following description of ζ (see Theorems VII.8.13 and VII.8.14 in the Complex Analysis notes):

Theorem VII.8.14. The zeta function is meromorphic in $\mathbb{C}$ with only a simple pole at $z = 1$ and $\mathrm{Res}(\zeta; 1) = 1$. For $z \neq 1$, ζ satisfies the Riemann Functional Equation:

$$\zeta(z) = 2(2\pi)^{z-1}\Gamma(1-z)\zeta(1-z)\sin\left(\frac{1}{2}\pi z\right).$$

Note/Definition. The gamma function $\Gamma(z)$ has simple poles at $z = 0, -1, -2, \ldots$ and is never 0. So $\Gamma(1-z)$ has simple poles at $z = 1, 2, 3, \ldots$. Now $\zeta(z)$ is analytic at $z = 2, 3, 4, \ldots$, so from Riemann's Functional Equation,

$$\zeta(z) = 2(2\pi)^{z-1}\Gamma(1-z)\zeta(1-z)\sin\left(\frac{1}{2}\pi z\right),$$

we have for $z = 2, 4, 6, \ldots$ that $\zeta(1-z)\sin(\pi z/2) = 0$ and the simple pole of $\Gamma(1-z)$ cancels with this zero for $z = 2, 4, 6, \ldots$ (otherwise $\zeta(z)$ would not be analytic at $z = 2, 4, 6, \ldots$). So $\zeta(z) \neq 0$ for $z = 2, 4, 6, \ldots$ (since the other factors of $\zeta(z)$ on the right-hand side of the functional equation are nonzero for these values of z). Now $\sin(\pi z/2) = 0$ for $z = -2, -4, -6, \ldots$ and $\Gamma(1-z)$ has no pole at these points, so $\zeta(z) = 0$ for $z = -2, -4, -6, \ldots$. The points $z = -2, -4, -6, \ldots$ are the *trivial zeros* of $\zeta(z)$. By the way, $\zeta(0) = -1/2$ so this covers all even integer values of z (where $\sin(\pi z/2)$ is 0). Any other zeros of $\zeta(z)$ are *nontrivial zeros*.

Note. Notice that Riemann's Functional Equation expresses $\zeta(z)$ in terms of $\zeta(1-z)$. Since $\Gamma(z)$ is never zero and the zeros of $\sin(\pi z/2)$ are addressed in the previous note, then the only other zeros $\zeta(z)$ must also be zeros of $\zeta(1-z)$ (and conversely). That is, $\zeta(z^*) = 0$ if and only if $\zeta(1-z^*) = 0$ where z^* is a nontrivial zero of ζ. So the nontrivial zeros form a set which is symmetric with respect to the vertical line $\text{Re}(z) = 1/2$.

Definition. The line $\text{Re}(z) = 1/2$ is the *critical line* of the Riemann zeta function.

Note. We are now in a position to state the Riemann Hypothesis. It concerns the location of the nontrivial zeros of ζ.

The Riemann Hypothesis.

The nontrivial zeros of the zeta function $\zeta(z)$ all lie on the critical line $\text{Re}(z) = 1/2$.

Note. It is easy to show that $\zeta(z) \neq 0$ for $\text{Re}(z) > 1$. For such z values we have the representation of $\zeta(z)$ as:

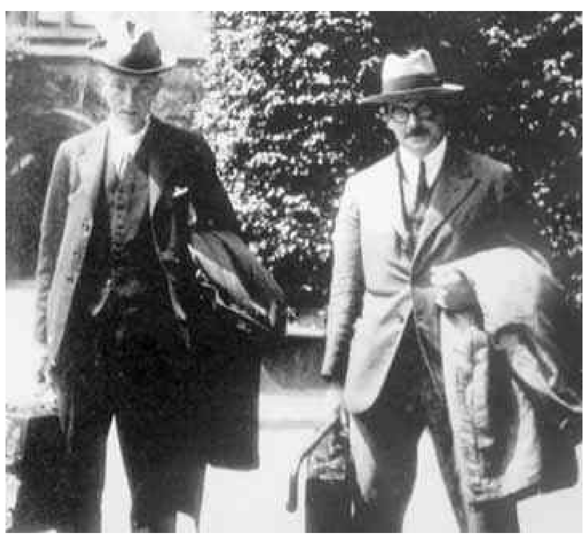
$$\zeta(z) = \sum_{n=1}^{\infty} \frac{1}{n^z} = \prod_{p \text{ prime}} \frac{1}{1 - p^{-z}} \text{ for } \text{Re}(z) > 1.$$

With $\text{Re}(z) > 1$, we can only have $\zeta(z) = 0$ if we have one of the factors on the right-hand side of this representation as 0. But $|p^{-z}| = p^{|\text{Re}(-1)|} = (1/p)^{|\text{Re}(z)|} \neq 1$ so that the right-hand side is nonzero for $\text{Re}(z) > 1$, and hence we now have that the zeros of $\zeta(z)$ lie in $\text{Re}(z) \leq 1$. But now from the functional equation (which was known to Riemann; see Edward's *Riemann's Zeta Function*, page 12), we have that for the nontrivial zeros that $\zeta(z) = 0$ if and only if $\zeta(1 - z) = 0$. So this implies that the nontrivial zeros of must satisfy $\text{Re}(1 - z) \leq 1$, or $\text{Re}(z) \geq 0$. We now see that the nontrivial zeros of ζ must lie in the vertical strip $0 \leq \text{Re}(z) \leq 1$ in the complex plane. As mentioned in my online notes on Supplement. The Prime

Number Theorem—History, Hadamard and de le Vallée Poussin proved the Prime Number Theorem by proving that ζ has no zeros on $\text{Re}(z) = 1$ (and hence, by the functional equation, no zeros on $\text{Re}(z) = 0$). So we can refine our observation on the location of the nontrivial zeros of ζ, so that we now see the region containing all of the nontrivial zeros of $\zeta(z)$ is the vertical strip $0 < \text{Re}(z) < 1$.

Definition. The vertical strip $0 < \text{Re}(z) < 1$ is the *critical strip* of the Riemann zeta function.

Note. We can now view the Riemann Hypothesis in a clearer context. We know that all the nontrivial zeros of ζ lie in the critical strip. The Riemann Hypothesis claims that they all lie on the center line of the critical strip, namely the critical line $\text{Re}(z) = 1/2$. An initial theoretical result concerning the Riemann Hypothesis (beyond initial computational results that determined some of the nontrivial zeros of ζ) is due to Godfrey Harold ("G.H.") Hardy (February 7, 1877–December 1, 1947) in 1914. Hardy proved that there are infinitely many zeros of ζ on the critical line in his "Sur les Zéros de la Fonction $\zeta(s)$ de Riemann," *C. R. Acad. Sci. Paris*, **158**, 1012-1014 (1914). Attention then turned to bounds on the *number of zeros* of ζ in various segments of the critical line. In 1921 Hardy and John E. Littlewood (June 9, 1885–September 6, 1977; a point of trivia: both Hardy and Littlewood are in my mathematical mathematical genealogy with, with Hardy as my mathematical great-great-great-grandfather and Littlewood as my mathematical great-great-great-great grandfather along another line—see my Mathematical Genealogy) proved that the number of roots on the line segment from $1/2$ to $1/2+iT$ is *at least* KT for some positive constant K and all sufficiently large T in "The Zeros of Riemann's Zeta-Function on the Critical Line," *Math. Z.*, **10**, 283–317 (1921). In 1942, Atle Selberg (June 14, 1917–August 6, 2007) who gave one of the elementary proofs of the Prime Number Theorem in 1949, proved that the number of such roots is at least $KT \log T$ for some positive constant K and all sufficiently large T. Selberg's work appeared in "On the Zeros of Riemann's Zeta-Function," *Skr. Norske Vid.-Akad. Oslo*, No. 10 (1942). Proofs of these three result appear in Chapter 11, "Zeros on the Line," of Harold M. Edwards' *Riemann's Zeta Function*, Pure and Applied Mathematics, A Series of Monographs and Textbooks, San Diego: Academic Press (1974); this book has also been in print by Dover Publications since 2001. An online copy of Edward's book is on the UCLA website (accessed 4/10/2022).



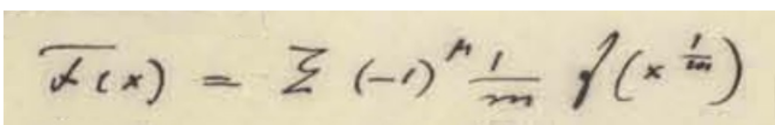
G. H. Hardy (left) and J. E. Littlewood (right)

Image from the MacTutor History of Mathematics Archive (accessed 4/10/2022).

Note. We have seen that the location of zeros of the Riemann zeta function was used in the proof of the Prime Number Theorem (namely, the fact that no zeros of ζ lie in $\text{Re}(z) \geq 1$). But why is it that the location of the other (nontrivial) zeros has become such a famous problem? The answer, again, relates to function $\pi(x)$ and the Prime Number Theorem. Riemann, in fact, not only proposes an approximation for $\pi(x)$, but he gives a formula for $\pi(x)$. (We follow the notation in Derbyshire's *Prime Obsession* here.) Riemann showed:

$$\pi(x) = \sum_{n=1}^{\infty} \frac{\mu(n)}{n} J(\sqrt[n]{x}). \qquad (*)$$

We'll soon see that for any given x the series is actually a finite sum, and we'll explain the functions μ and J. Here is this equation in Riemann's own writing. This is a close-up of a manuscript for the 1859 paper and can be found on the Clay Mathematics Institute webpage (accessed 4/10/2022).



Riemann represents the function $\pi(x)$ with a capital script F, he sums over m, and he represents the function $\mu(n)$ in a way that agrees with the definition we give below, so that Riemann's $(-1)^\mu$ is the same as our $\mu(n)$. Also, Riemann uses a lower case script f where we use the symbol J.

Note/Definition. The Möbius function is defined in my online notes for Mathematical Reasoning (MATH 3000) on Section 6.9. Perfect Numbers, Mersenne Primes, Arithmetic Functions as the arithmetic function $\mu : \mathbb{N} \to \{-1, 0, 1\}$ given by

$$\mu(n) = \begin{cases} 1 & \text{if } n = 1 \\ (-1)^r & \text{if } n = p_1 p_2 \cdots p_r, \text{ a product of } r \text{ distinct primes} \\ 0 & \text{otherwise.} \end{cases}$$

Note/Definition. The J function is defined in terms of the π function as:

$$J(x) = \pi(x) + \frac{1}{2}\pi((\sqrt{x}) + \frac{1}{3}\pi(\sqrt[3]{x}) + \frac{1}{4}\pi(\sqrt[4]{x}) + \frac{1}{5}\pi(\sqrt[5]{x}) + \cdots .$$

See Expression 19-1 on page 299 of Derbyshire. Notice that for every $x \geq 1$ we have $\sqrt[n]{x} \to 1$ as $n \to \infty$, and $\pi(x) = 0$ for $x < 2$ (since 2 is the least prime number). So for fixed $x \geq 1$ and n sufficiently large, we have $\pi(\sqrt[n]{x}) = 0$ and we see that in fact $J(x)$ is represented as a finite sum. In Riemann's 1859 manuscript, you can see his statement of this definition of J just above his statement of $\pi(x)$ (or "$F(x)$," as he state it) shown above. In fact, it is from the definition of J in terms of π that Riemann derives the formula $(*)$ of π in terms of J. The process by which this derivation occurs is called "Möbius inversion" (see Derbysire, page 302). The reason for this "change in direction" is that because Riemann has a way to express J in terms of ζ.

Note. The precise formula for j in terms of the zeta function is

$$J(x) = \mathrm{Li}(x) - \sum_\rho \mathrm{Li}(x^\rho) - \log 2 + \int_x^\infty \frac{dt}{t(t^2-1)\log t},$$

where the summation is taken over all ρ which are zeros of the zeta function. So **there** is the connection between the zeros of ζ and $\pi(x)$!!! Notice form equation

$(*)$ we have

$$\pi(x) = \sum_{n=1}^{\infty} \frac{\mu(n)}{n} J(\sqrt[n]{x}) = \sum_{n=1}^{\infty} \frac{\mu(n)}{n} \left(\mathrm{Li}(x) - \sum_{\rho} \mathrm{Li}(x^{\rho}) - \log 2 + \int_x^{\infty} \frac{dt}{t(t^2-1)\log t} \right).$$

This results in four terms on the right-hand side of the equation:

1. The principal term $\frac{\mu(n)}{n}\mathrm{Li}(x)$,

2. The secondary term (called the "periodic term" by Riemann) $-\frac{\mu(n)}{n}\sum_{\rho}\mathrm{Li}(x^{\rho})$,

3. The $\log 2$ term $\frac{\mu(n)}{n}\log 2$, and

4. The integral term $\frac{\mu(n)}{n}\int_x^{\infty}\frac{dt}{t(t^2-1)\log t}$.

The $\log 2$ and integral terms are "negligible"; see Derbyshire pages 343 to 345 where he introduces this terminology, and his Chapter 21 in which he goes through some specific computations to illustrate the ideas and the inversion.

Note. According to Brian Conrey, of the American Institute of Mathematics and the University of Bristol, in his online Riemann's Hypothesis document, see page 5 (accessed 4/12/2022):

> "Thus the difference between Riemann's formula and Gauss' conjecture is, to a first estimation, about $\mathrm{Li}(x^{\beta_0})$ where β_0 is the largest or the supremum of the real parts of the zeros. Riemann conjectured that all of the zeros have real part $\beta = 1/2$ so that the error term is of size $x^{1/2}\log x$. This assertion of the perfect balance of the zeros, and so of the primes, is Riemann's Hypothesis."

The idea of the "perfect balance" comes from the fact that the nontrivial zeros of ζ are symmetric with respect to the critical line $\text{Re}(z) = 1/2$ (by the functional equation). So if $\beta_0 = 1/2$, the error term $\text{Li}(x^{1/2})$ is minimized; any nontrivial zero off of the critical line either has real part greater than $1/2$ or has a symmetric zero with real part greater than $1/2$.

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