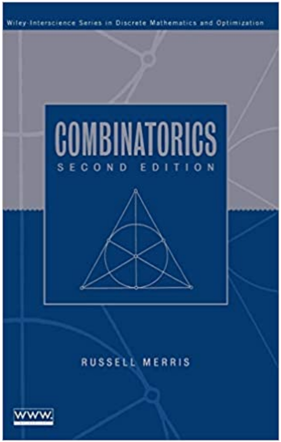
Applied Combinatorics and Problem Solving

Chapter 1. The Mathematics of Choice

1.1. The Fundamental Counting Principle—Proofs of Theorems



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Exercise 1.1.8(b)

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$1 \times 1! + 2 \times 2! + 3 \times 3! + \cdots + n \times n! = (n+1)! - 1.$

Proof. Notice that by Exercise 1.1.2(c) we have $(n+1) \times (n!) = (n+1)!$.
Repeatedly applying this, we have

$$\begin{aligned}
(n+1)! - 1 &= (n+1) \times n! - 1 \text{ by Exercise 1.1.2(c)} \\
&= n \times n! + n! - 1 \\
&= n \times n! + (n) \times (n-1)! - 1 \text{ by Exercise 1.1.2(c)} \\
&= n \times n! + (n-1+1) \times (n-1)! - 1 \\
&= n \times n! + (n-1) \times (n-1)! + (n-1)! - 1 \\
&= n \times n! + (n-1) \times (n-1)! + (n-1) \times (n-2)! - 1 \\
&\quad \text{by Exercise 1.1.2(c)} \\
&= n \times n! + (n-1) \times (n-1)! + (n-2+1) \times (n-2)! - 1
\end{aligned}$$

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Exercise 1.1.8(b), continued

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Proof (continued). Notice that by Exercise 1.1.2(c) we have $(n+1) \times (n!) = (n+1)!$. Repeatedly applying this, we have

$$\begin{aligned} (n+1)! - 1 &= n \times n! + (n-1) \times (n-1)! + (n-2) \times (n-2)! \\ &\quad + (n-2)! - 1 \\ &\ \vdots \\ &= n \times n! + (n-1) \times (n-1)! + (n-2) \times (n-2)! + \cdots \\ &\quad + 3 \times 3! + 2 \times 2! + 1 \times 1! + 0! - 1 \\ &= n \times n! + (n-1) \times (n-1)! + (n-2) \times (n-2)! + \cdots \\ &\quad + 3 \times 3! + 2 \times 2! + 1 \times 1!, \end{aligned}$$

as claimed. (Note: We could also give an inductive proof.) □

Exercise 1.1.22

Exercise 1.1.22. In how many different ways can eight coins be arranged on an 8×8 checkerboard so that no two coins lie in the same row or column?

Proof. Number the columns 1 through 8. Let c_i be the number of choices for a row in which to put a coin in column i for $1 \le i \le 8$. In Column 1, the coin can go in any of the 8 rows so that $c_1 = 8$. In Column 2, the coin can go in any of the rows, except the row used with the first coin so that $c_2 = 7$. Similarly, $c_3 = 6$, $c_4 = 5$, $c_5 = 4$, $c_6 = 3$, $c_7 = 2$, and $c_8 = 1$. So by the Fundamental Counting Principle, the number of ways to arrange the coins is

$$c_1 c_2 c_3 c_4 c_5 c_6 c_7 c_8 = 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 8! = \boxed{40{,}320}. \quad \square$$

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