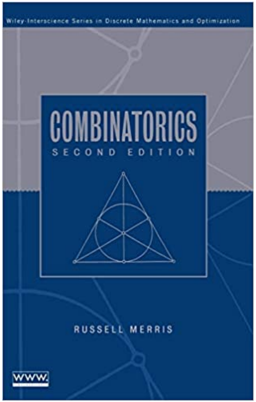
Applied Combinatorics and Problem Solving

Chapter 1. The Mathematics of Choice

1.2. Pascal's Triangle—Proofs of Theorems



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If $1 \leq r \leq n$, then $C(n+1, r) = C(n, r-1) + C(n, r)$.

Proof. Let S be the $(n+1)$-element set $S = \{x_1, x_2, \ldots, x_n, y\}$. Its r-element subsets, of which there are $C(n+1, r)$, can be partitioned into two families, those that contain y and those that do not.

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Theorem 1.2.6

Theorem 1.2.6. The rows of Pascal's triangle are unimodal. That is, the numbers in each row increase from left to right, up to the middle of the row and then decrease from the middle to the right-hand end.

Proof. If $n > 2r + 1$, then the ratio

$$\frac{C(n, r+1)}{C(n, r)} = \frac{r!(n-r)!n!}{(r+1)!(n-r-1)!n!} = \frac{n-r}{r+1} > 1,$$

so that $C(n, r+1) > C(n, r)$ for $r < (n-1)/2$. That is the numbers in each row increase from left to right up to the middle of the row. By the symmetry property, $C(n, r) = C(n, n-r)$, the row decreases from the middle to the right-hand end (that is, for $r > (n-1)/2$), as claimed. □

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Exercise 1.2.18

Exercise 1.2.18. Packing for a vacation, a young man decides to take 3 long-sleeve shirts, 4 short-sleeve shirts, and 2 pairs of pants. If he owns 16 long-sleeve shirts, 20 short-sleeve shirts, and 13 pairs of pants, in how many different ways can he pack for the trip?

Proof. We consider the ways to choose the long-sleeve shirts, short-sleeve shirts, and the pants. By Note 1.2.C, we can choose $r = 3$ long-sleeve shirts from a set of $n = 16$ long-sleeve shirts in $C(16,3) = \dfrac{16!}{13!3!} = \dfrac{16 \times 15 \times 14}{3 \times 2 \times 1} = 560$ ways. Similarly, we can choose $r = 4$ short-sleeve shirts from a set of $n = 20$ short-sleeve shirts in $C(20,4) = \dfrac{20!}{16!4!} = \dfrac{20 \times 19 \times 18 \times 17}{4 \times 3 \times 2 \times 1} = 4845$ ways. Also, we can choose $r = 2$ pairs of pants from a set of $n = 13$ pairs of pants in $C(13,2) = \dfrac{13!}{11!2!} = \dfrac{13 \times 12}{2 \times 1} = 78$ ways.

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Exercise 1.2.26

Exercise 1.2.26. Let S be an n-element set, where $n \geq 1$. If A is a subset of S, denote by $o(A)$ the *cardinality* of (number of elements in) A. Say that A is odd (even) if $o(A)$ is odd (even). Prove that the number of odd subsets of S is equal to the number of its even subsets.

Proof. Let $x \in S$. We partition the subsets of S into two categories: (1) category C_1 of those that contain element x, and (2) category C_2 of those that do not contain element x. We set up a correspondence between the odd sets in category C_1 with the even sets in category C_2, and a correspondence between the even sets in category C_1 with the odd sets in category C_2.

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For each A a subset of S with A odd and in category C_1, there is the corresponding even set in category C_2; namely, the set $A \setminus \{x\}$. Conversely, for each even set B in category C_2, there is the corresponding odd set $A = B \cup \{x\}$ in category C_1. So the number of odd sets in category C_1 is the same as the number of even sets in category C_2.

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Since categories C_1 and C_2 are disjoint and union to give the set of all subsets of S (that is, $\mathcal{P}(S) = C_1 \cup C_2$), then by the Second Counting Principle (Principle 1.2.3) the number of even subsets of S equals the number of odd subsets of S, as claimed. □

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