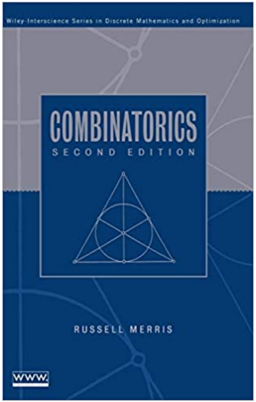
Applied Combinatorics and Problem Solving

Chapter 1. The Mathematics of Choice

1.3. Elementary Probability—Proofs of Theorems



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Theorem 1.3.5

Theorem 1.3.5. Let E be a fixed but arbitrary sample space. If A and B are subsets of E, then

$$P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B).$$

Proof. The number of elements in $A \cup B$ is the number of elements in A plus the number of elements in B minus the number of elements in both A and B (that is, minus $o(A \cap B)$), so that

$$o(A \cup B) = o(A) + o(B) - o(A \cap B).$$

(This is a special case of the Principle of Inclusion and Exclusion, to be seen in Chapter 2.) Dividing both sides of this equation by $o(E)$ gives

$$\frac{o(A \cup B)}{o(E)} = \frac{o(A)}{o(E)} + \frac{o(B)}{o(E)} - \frac{o(A \cap B)}{o(E)}$$

$$= P(A) + P(B) - P(A \cap B) = P(A) + P(B) - P(A \text{ and } B),$$

by Definition 1.3.4. □

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Corollary 1.3.6

Corollary 1.3.6. Let E be a fixed but arbitrary sample space. If A and B are subsets of E, then $P(A \text{ or } B) \leq P(A) + P(B)$, with equality if and only if A and B are disjoint.

Proof. Since $P(A \text{ and } B) \geq 0$, then the inequality follows from Theorem 1.3.5. Equality holds (by Theorem 1.3.5) if and only if $P(A \text{ and } B) = 0$, which holds if and only if $o(A \cap B) = 0$ or if and only if $A \cap B = \varnothing$ (i.e., A and B are disjoint), as claimed. □

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Theorem 1.3.11

Theorem 1.3.11. Let E be a fixed but arbitrary sample space. If A and B are subsets of E, then

$$P(A \text{ and } B) = P(A)P(B|A).$$

Proof. Let $D = o(E)$, $a = o(A)$, and $N = o(A \cap B)$. If $a = 0$, both sides of the claimed equation are 0 and the claim holds. Otherwise, $P(A) = a/D$, $P(B|A) = N/a$ (by Definition 1.3.10), and

$$P(A)P(B|A) = (a/D)(N/a) = N/D = P(A \cap B) = P(A \text{ and } B),$$

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as claimed. □

Corollary 1.3.12

Corollary 1.3.12. Bayes' First Rule.
Let E be a fixed but arbitrary sample space. If A and B are subsets of E, then $P(A)P(B|A) = P(B)P(A|B)$.

Proof. By Theorem 1.3.11, $P(A \text{ and } B) = P(A)P(B|A)$ and (interchanging A and B) $P(B \text{ and } A) = P(B)P(A|B)$. Of course $P(A \text{ and } B) = P(B \text{ and } A)$, so that

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