1.1. The Fundamental Counting Principle

Note. According to ETSU's own Dr. Robert Beeler, "Put simply, *combinatorics* is the mathematics of counting." This is the first sentence in the first section of his *How to Count: An Introduction to Combinatorics and Its Applications* (Springer, 2015). In this section, we state and apply the Fundamental Counting Principle. We will use this idea throughout the course (so often that, after some experience, we will not even reference it).

Note. You will encounter the Fundamental Counting Principle in several of your other classes. For example, you are likely to see it in:

Foundations of Probability and Statistics-Calculus Based (MATH 2050). See my online notes for this class on Section 2.2. Counting Methods,

Mathematical Reasoning (MATH 3000). See my online notes on Section 4.1. Cardinality; Fundamental Counting Principles (notice Theorem 4.7 which expresses the counting principle in terms of the size of a Cartesian product of sets),

Graduate Level Combinatorics. This is not an official ETSU class (but easily could be). See my notes on Section 1.1. The Sum and Product Rules for Sets (notice Lemma 1.1.1(b) where, again, the counting principle is stated in terms of Cartesian products; the Fundamental Counting Principle called the "Product Rule" in this setting).

Note. The **Fundamental Counting Principle** states:

> Consider a (finite) sequence of decisions. Suppose the number of choices for each individual decision is independent of decisions made previously in the sequence. Then the number of ways to make the whole sequence of decisions is the product of these numbers of choices.

Symbolically, suppose c_i is the number of choices for decision i, where $1 \leq i < n$. Assuming the number of choices c_{i+1} does not depend on the previous choices made for $1, 2, \ldots, i$, then the number of different ways to make the sequence of decisions is $c_1 \times c_2 \times \cdots \times c_n$.

Note. Consider the letters L, U, C, K. We can use the Fundamental Counting Principle to count the number of different four-letter words we can make by using these letters (each one exactly once in a word). By making an exhaustive list, we see that there are 24 possible words; see Figure 1.1.1.

LUCK	LUKC	LCUK	LCKU	LKUC	LKCU
ULCK	ULKC	UCLK	UCKL	UKLC	UKCL
CLUK	CLKU	CULK	CUKL	CKLU	CKUL
KLUC	KLCU	KULC	KUCL	KCLU	KCUL

Figure 1.1.1. The rearrangements of LUCK.

Note. We now apply the Fundamental Counting Principle to the number of of words one can make from letters L, U, C, K. There are four decisions to be made: (1) decide on the first letter, (2) decide on the second letter, (3) decide on the third letter, and (4) decide on the fourth letter. With the c_i notation mentioned in the statement of the Fundamental Counting Principle, we have $c_1 = 4$ ways to choose the first letter (i.e., to make the first decision), $c_2 = 3$ ways to choose the second letter (it can be any of the four letters, other than the first one chosen), $c_3 = 2$ ways to choose the third letter (it can be any of the four letters, other than the first two), and $c_4 = 1$ ways to choose the fourth letter (after the first three decisions, there is only one choice for the last letter). So the number of ways the decisions can be made (i.e., that number of words that can be made) is $c_1 \times c_2 \times c_3 \times c_4 = 4 \times 3 \times 2 \times 1 = 24$, as observed in Figure 1.1.1.

Definition. For positive integer n, define *n-factorial* as $n! = n \times (n-1) \times \cdots 2 \times 1$. Define $0! = 1$.

Note. We see that the number of four-letter words that can be made with the letters L, U, C, K is $4!$. It can similarly be shown that the number of n-letter words that can be made from n different letters (with the same rules that each letter is used only once in each word) is $n!$.

Note. Now consider the number of four-letter words that can be made from the letters L, O, O, T. This time, we repeat letters. If we were to subscript the O's and considered then the four *different* letters L, O_1, O_2, T, then we know that there are $4! = 24$ different words that result. Now if we ignore the subscripts on the O's we have words repeated in the list of 24. For example, words $\mathrm{LO_1TO_2}$ and $\mathrm{LO_2TO_1}$ both yield the word LOTO when the subscripts are ignored. In this way, we see that the list of 24 words repeats every word twice. So the number of different words that can be made with the letters L, O, O, T are $4!/2 = 12$.

Note. Next, consider the number of four-letter words that can be made from the letters L, U, L, L. If we again add subscripts to the repeated letters and consider the letters L_1, U, L_2, L_3 then we get $4! = 24$ different words. This ignoring the subscripts yields the same word several times. With the subscripts, L_1,L_2, and L_3 occur is some positions in the word. We can freely interchange these letters among these positions, and when the subscripts are dropped we end up ultimately with the same word using the letter L, U, L, L. In these three positions, we can have any of L_1, L_2, L_3 in the first position, and of the remaining two letters in the second position, and the final remaining subscripted L must be in the third position. So there are $3! = 6$ words based on L_1, U, L_2, L_3 for each word based on L, U, L, L. For example, the word LULL corresponds to the words $\mathrm{L}_1\mathrm{U}\mathrm{L}_2\mathrm{L}_3$, $\mathrm{L}_1\mathrm{U}\mathrm{L}_3\mathrm{L}_2$, $\mathrm{L}_2\mathrm{U}\mathrm{L}_1\mathrm{L}_3$, $\mathrm{L}_2\mathrm{U}\mathrm{L}_3\mathrm{L}_1$, $\mathrm{L}_3\mathrm{U}\mathrm{L}_1\mathrm{L}_2$, and $\mathrm{L}_3\mathrm{U}\mathrm{L}_2\mathrm{L}_1$. Therefore, the number of different words that an be made with the letters L, U, L, L is $4!/6 = 4$. These four words are LLLU, LLUL, LULL, and ULLL.

Note. For a related problem, consider the letters M, I, S, S, I, S, S, I, P, P, I. Here there are 11 letters, with four I's, four S's, two P's, and one M. Again, we could subscript the letters and calculate that there are 11! resulting words. The fact that we can interchange P_1 and P_2 when ignoring the subscripts, we see that 11! is an overcount by a factor of 2 with respect to the P's. Similarly, 11! is an overcount by a factor of 4! with respect to both the I's and the S's. So the total number of different words that can be made from the 11 letters is $\dfrac{11!}{4!4!2!1!} = 34{,}650$ (we include the factor of 1! for the letter M to better see the pattern. This inspires the next definition.

Definition 1.1.2. The *multinomial coefficient* is $\displaystyle\binom{n}{r_1, r_2, \ldots, r_k} = \frac{n!}{r_1! r_2! \cdots r_k!}$, where $r_1 + r_2 + \cdots r_k = n$.

Note 1.1.A. The number of different words of length n that can be made from a collection of letters, where there are k different letters with repetitions of $r_1, r_2, \ldots, r_k$ times (so that $r_1 + r_2 + \cdots r_k = n$), is $\displaystyle\binom{n}{r_1, r_2, \ldots, r_k} = \frac{n!}{r_1! r_2! \cdots r_k!}$.

Note. Merris includes a discussion of the POSTNET ("Postal Numeric Encoding Technique") barcodes that are printed at the bottom of letters that go through the United States Postal Service. The numerals 0 through 9 are each represented by five vertical lines where three are short lines and two are long lines. So we have words of length five consisting of two letters, one is repeated three times and the other is repeated twice. In the notation above we have $n = 5$, $k = 2$, $r_1 = 3$, and $r_2 = 2$. So the number of "words" we can make is $\displaystyle\binom{n}{r_1, r_2, \ldots, r_k} = \frac{5!}{3!2!} = \frac{120}{12} = 10$, as desired. The representations of 0 through 9 are given in Figure 1.1.3. (You might see a hidden use of the idea of binary-type of representation in this figure, but you need to think of 0 as following 9 for this interpretation to be consistent.)

0 = 1 = 2 = 3 = 4 =

5 = 6 = 7 = 8 = 9 =

Figure 1.1.3. POSTNET barcodes.

What is printed on an envelope is 52 vertical lines. The first and last vertical lines are long and are not part of the encoded number. Then the 5-digit ZIP code is given, followed by the "+4 (often representing a box number), and finally a check digit. The check digit is chosen so that the sum of the total of ten digits is a multiple of 10. So the 52 vertical lines correspond to the two large vertical lines on the end and $5 \times 10 = 50$ vertical lines that represent the ZIP+4 and the check digit. For example, the ETSU Department of Mathematics and Statistics has a ZIP+4 code of 37614-0663 (the department's box number is 70663). Since $3+7+6+1+4+0+6+6+3 = 36$ then the check digit is 4. So the POSTNET barcode will represent 3761406634 and is given in the next figure (with an explanation in color).

3 7 6 1 4 0 6 6 3 4

Example 1.1.4. If we can write a given positive integer as a product of powers of prime numbers (which can be done in a unique way by the Fundamental Theorem of Arithmetic; see either my online notes for Mathematical Reasoning (MATH 3000) on Section 6.3. Divisibility: The Fundamental Theorem of Arithmetic, or notes for Elementary Number Theory (MATH 3120) on Section 2. Unique Factorization), then we can use this factorization and the Fundamental Counting Principle to find the number of divisors of the positive integer. For example, for $n = 360$ we have $360 = 2^3 \times 3^2 \times 5$. If d is a divisor of 360, then by the Fundamental Theorem of Arithmetic we must have $d = 2^a \times 3^b \times 5^c$ where $0 \leq a \leq 3$, $0 \leq b \leq 2$, and $0 \leq c \leq 1$. So to get a divisor, we choose the exponents a, b, c. There are four choices for a, three choices for b, and two choices for c. Hence, by the Fundamental Counting Principle there are $4 \times 3 \times 2 = 24$ choices for the exponents and hence 24 resulting divisors of 360.

Exercise 1.1.8(b). Prove that $1 \times 1! + 2 \times 2! + 3 \times 3! + \cdots + n \times n! = (n+1)! - 1$.

Exercise 1.1.22. In how many different ways can eight coins be arranged on an 8×8 checkerboard so that no two coins lie in the same row or column?

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