1.2. Pascal's Triangle

Note. In the previous section we considered the number of words that can be made from different collections of letters. In a word, the order of the letters matter! In this section we consider collections in which order does not matter (such as when you are dealt a hand of cards; the order in which you were dealt the cards does not matter, all that matters is the cards which you hold). We count these combinations of objects.

Note/Definition. Consider a collection of n elements in set S. We want to count the number of ways to choose r-element subsets of S. Denote this quantity as $C(n,r)$, read "n-choose-r." Two values of this quantity are clear: $C(n,n) = 1$ and $C(n,1) = n$. That is, there is one way to choose all n elements of S, and there are n ways to choose a single element of set S (since S has n elements). Notice that choosing a subset of S of size r is equivalent to choosing a subset of size $n-r$; in the first case you are choosing r elements to "take" and $n-r$ elements to "leave," and in the second case you are choosing $n-r$ elements to take and r elements to leave. But when the elements to take are chosen, then the elements to leave are determined. Because of this, parameter $C(n,r)$ has the symmetry property: $C(n,r) = C(n,n-r)$. Notice that this property and the fact that $C(n,n) = 1$ implies that $C(n,0) = 1$; of course there is only one way to choose none of the elements of S, namely to choose the null set $\varnothing$. By convention, we take $C(0,0) = 1$. Before we state our first result concerning $C(n,r)$, we need another counting principle.

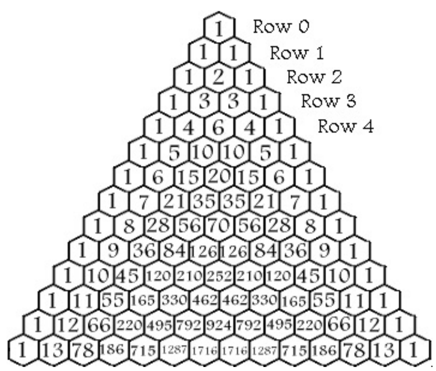
Principle 1.2.3. The Second Counting Principle.

If a set can be expressed as the disjoint union of two (or more) subsets, then the number of elements in the set is the sum of the numbers of elements in the subsets.

Theorem 1.2.2. Pascal's Relation.

If $1 \le r \le n$, then $C(n+1, r) = C(n, r-1) + C(n, r)$.

Note/Definition. Merris presents three tables values of $C(n, r)$, each in a triangular pattern (see pages 12 and 13). Instead, we consider the following image.



From Pascal's Triangle and its Patterns (accessed 3/25/2022).

This collection of hexagons contains the values of $C(n, r)$ in the pattern:

$$\begin{array}{ccccccc} & & & C(0,0) & & & \\ & & C(1,0) & & C(1,1) & & \\ & C(2,0) & & C(2,1) & & C(2,2) & \\ C(3,0) & & C(3,1) & & C(3,2) & & C(3,3) \\ \iddots & & \vdots & & \vdots & & \ddots \end{array}$$

In each hexagon, notice that the entry is the sum of the entries in the adjoining hexagons to the upper left and upper right (this is Pascal's Relation); this gives a geometric interpretation of Pascal's Relation (Theorem 1.2.2). For example, in Row 4 we have an entry of 6, with entries of 3 to both the upper left and upper right. This corresponds to the relationship $C(4,2) = C(3,1) + C(3,2)$. Notice that this follows from Pascal's Relation with $n = 3$ and $r = 2$. The arrangements above are called *Pascal's triangle*.

Note 1.2.A. An additional observation is the value of the sum of each row of Pascal's triangle. Taking the sum of the values in each row (starting with Row 0) we get: 1, $1+1=2$, $1+2+1=4$, $1+3+3+1=8$, $1+4+6+4+1=16$, $1+5+10+10+5+1=32$, etc. It appears that the sum of a row is a power of 2; in fact, the sum of the entries in Row n appears to be 2^n. Summing along Row n gives us the number of all subsets of n-element set S,

$$C(n,0) + C(n,1) + \cdots + C(n,n-1) + C(n,n) = \sum_{r=0}^{n} C(n,r),$$

since $C(n,r)$ is the number of subsets set S of size r and we letting r range over all of the possible sizes of a subset of S here. Now we can count the number of subsets of $S = x_1, x_2, \ldots, x_n\}$ by considering which elements are and which are not in a possible subset. We can go through the elements one at a time and make a decision of (1) include, or (2) exclude that element in subset. So for element x_1 there are 2 possible decisions, for element x_2 there are two possible decisions, $\ldots$, for element x_n there are two possible decisions. So, by the Fundamental Counting Principle, there are $2 \times 2 \times \cdots \times 2 = 2^n$ possible subsets of set S.

Note. You are likely familiar with the idea that a set of cardinality n has 2^n subsets. You see this in Mathematical Reasoning (MATH 3000) where the *power set* of a given set S is defined to be the set of all subsets of S, denoted $\mathcal{P}(S)$; see my online notes for Mathematical Reasoning on Section 2.10. Mathematical Induction and Recursion (notice Theorem 2.69).

Note 1.2.B. Related to Note 1.2.A, we can construct a word of length n based on the letters Y and N which is associated with each subset of $S = \{x_1, x_2, \ldots, x_n\}$. We described the construction of a subset of S and a sequence of decisions, which we can reword as (1) YES include element x_i in the subset, or (2) NO do not include x_i in the subset. For example the word on length n of all N's corresponds to the subset $\varnothing$ of set S. The word of length n of all Y's corresponds to the subset S itself. Each such word corresponds to a unique subset of S, and each subset of S corresponds to a unique such word. So these are equivalent approaches to the same question. Now such a word of length n that contains r Y's (where $0 \le r \le n$), and hence $n = r$ N's, corresponds a subset of S that contains r elements. We know from Note 1.1.A, that in terms of the multinomial coefficient there are $\binom{n}{r, n-r} = \frac{n!}{r!(n-r)!}$ words on the two letters Y and N that contain r Y's and $n-r$ N's. Since this is the same as the number of subsets of S of a set of size r, then we have the following algebraic representation of $C(n, r)$:

$$C(n, r) = \frac{n!}{r!(n-r)!}.$$

Note 1.2.C. An alternative notation for n-choose-r is $C(n,r) = \binom{n}{r}$. It is also commonly referred to as the "number of *combinations* of r objects from a set of size n." This is often the case in a statistics setting; see my online notes for Foundations of Probability and Statistics-Calculus Based (MATH 2050) on Section 2.2. Counting Methods (notice Note 2.2.D). You may have also seen $C(n,r)$ referred to as the *binomial coefficients* (consistent with the "multinomial coefficient" term used in Section 1.1. This is because when we raise the binomial $(x+y)$ to the nth power we get:

$$(x+y)^n = \sum_{r=0}^{n} C(n,r)x^{n-r}y^r.$$

See my online notes for Discrete Structures (MATH 2710) on Section 4.3. Permutations and Combinations; notice Theorem 4.3.6, The Binomial Theorem, which is to be proved by induction. Notice that with $x = y = 1$, the Binomial Theorem gives $2^n = \sum_{r=0}^{n} C(n,r)$, as observed above in Note 1.2.A.

Example 1.2.4. Consider a standard deck of 52 cards (with no jokers). In five-card poker, one is dealt 5 cards. How many possible hands of 5 cards are there?

Solution. We are interested in a subset of size $r = 5$ from a set S of size $n = 52$. So the number of such subsets (and hence the number of hands of 5 cards from a deck of 52) is

$$C(52,5) = \frac{52!}{5!(52-5)!} = \frac{52!}{5!47!} = \frac{52 \times 51 \times 50 \times 49 \times 48 \times 47!}{5! \times 47!}$$
$$= \frac{52 \times 51 \times 50 \times 49 \times 48}{5 \times 4 \times 3 \times 2 \times 1} = 2{,}598{,}960. \qquad \square$$

Theorem 1.2.6. The rows of Pascal's triangle are unimodal. That is, the numbers in each row increase from left to right, up to the middle of the row and then decrease from the middle to the right-hand end.

Exercise 1.2.18. Packing for a vacation, a young man decides to take 3 long-sleeve shirts, 4 short-sleeve shirts, and 2 pairs of pants. If he owns 16 long-sleeve shirts, 20 short-sleeve shirts, and 13 pairs of pants, in how many different ways can he pack for the trip?

Exercise 1.2.26. Let S be an n-element set, where $n \geq 1$. If A is a subset of S, denote by $o(A)$ the *cardinality* of (number of elements in) A. Say that A is odd (even) if $o(A)$ is odd (even). Prove that the number of odd subsets of S is equal to the number of its even subsets.