1.5. Combinatorial Identities

Note. In this section, we relate the binomial coefficients $C(n,r) = \binom{n}{r}$ (see Note 1.2.C) to the multinomial coefficients $\binom{n}{r_1, r_2, \ldots, r_k}$ where $r_1 + r_2 + \cdots + r_k = n$ (see Definition 1.1.2). We use these relationships to find a formula for the sum of the mth power of the first n natural numbers.

Note. As we saw in Note 1.2.B, the binomial coefficient $C(n,r)$ is a special case of the multinomial coefficients, since $C(n,r) = \dfrac{n!}{r!(n-r)!} = \binom{n}{r, n-r}$. The next result expresses the general multinomial coefficient in terms of binomial coefficients.

Theorem 1.5.1. If $r_1 + r_2 + \cdots r_k = n$, then

$$\binom{n}{r_1, r_2, \ldots, r_k} = \binom{n}{r_1}\binom{n-r_1}{r_2}\binom{n-r_1-r_2}{r_3}\cdots\binom{n-r_1-r_2-\cdots-r_{k-1}}{r_k}.$$

Note. We can also use summations to relate certain binomial coefficients to sums of other binomial coefficients, as follows.

Theorem 1.5.2. Chu's Theorem.

If $n \geq r$, then

$$\sum_{k=0}^{n} C(k,r) = C(r,r) + C(r+1,r) + C(r+2,r) + \cdots + C(n,r) = C(n+1,r+1)$$

(where $\sum_{k=0}^{n} C(k,r) = \sum_{k=r}^{n} C(k,r)$ because $C(k,r) = 0$ for $k < r$).

Note. In Calculus 1 (MATH 1910), you encounter the following formulae when you are introduced to Riemann integration:

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2} \quad \sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6} \quad \sum_{k=1}^{n} k^3 = \left(\frac{n(n+1)}{2}\right)^2.$$

(See my online Calculus 1 notes on Section 5.2. Sigma Notation and Limits of Finite Sums, and notice Theorem 5.2.B.) These equations are commonly proved using mathematical induction. For example, the first formula is proved inductively in Mathematical Reasoning (MATH 3000); see my online notes for Mathematical Reasoning on Section 2.10. Mathematical Induction and Recursion, and notice Example 2.67. Notice that $\sum_{k=1}^{n} k = \frac{n(n+1)}{2} = C(n+1,2)$. Also, we can express $k^2 = k + k(k-1) = C(k,1) + 2C(k,2)$ so that we can conclude

$$\sum_{k=1}^{n} k^2 = \sum_{k=1}^{n} C(k,1) + 2\sum_{k=1}^{n} C(k,2).$$

By Chu's Theorem (Theorem 1.5.2) we have (with $r=1$ and $r=2$):

$$\sum_{k=0}^{n} C(k,1) = C(n+1,2) \text{ and } \sum_{k=0}^{n} C(k,2) = C(n+1,3),$$

and, as expected, we have

$$\begin{aligned}\sum_{k=1}^{n} k^2 &= \sum_{k=1}^{n} C(k,1) + 2\sum_{k=1}^{n} C(k,2) = C(n+1,2) + 2C(n+1,3)\\ &= \frac{n(n+1)}{2} + 2\frac{(n+1)n(n-1)}{6} = n(n+1)\left(\frac{1}{2} + \frac{2(n-1)}{6}\right)\\ &= n(n+1)\frac{3+2(n-1)}{6} = \frac{m(n+1)(2n-1)}{6}.\end{aligned}$$

Note. For $m=2$, we have in the previous note that $\sum_{k=1}^{n} k^2 = \sum_{k=1}^{n} C(k,1) + 2\sum_{k=1}^{n} C(k,2)$. *If* for general $m \in \mathbb{N}$ we could find a formula of the form $k^m =$

$\sum_{r=1}^{m} a_{r,m} C(k,r)$ (where the $a_{r,m}$ are independent of k), then we could express the sum of the mth powers of k as

$$\sum_{k=1}^{n} k^m = \sum_{k=1}^{n} \sum_{r=1}^{m} a_{r,m} C(k,r) = \sum_{r=1}^{m} a_{r,m} \sum_{k=1}^{n} C(k,r) = \sum_{r=1}^{m} a_{r,m} C(n+1, r+1),$$

where the last equality holds by Chu's Theorem (Theorem 1.5.2). In fact, we will have a formula for $k^m = \sum_{r=1}^{m} a_{r,m} C(k,r)$ below (see Theorem 1.5.5) where the $a_{r,m}$ themselves are expressed in terms of binomial coefficients. To further illustrate this idea, consider the case of $m = 3$. We need the coefficients $x = a_{1,3}$, $y = a_{2,3}$, and $z = a_{3,3}$ such that

$$\begin{aligned} k^3 = a_{1,3} C(k,1) + a_{2,3} C(k,2) + a_{3,3} C(k,3) &= xC(k,1) + yC(k,2) + zC(k,3) \\ &= xk + y\frac{k(k-1)}{2} + z\frac{k(k-1)(k-2)}{6}. \end{aligned}$$

Collecting together the powers of k, we have

$$(z-6)k^3 + (3y-3z)k^2 + (6x-3y+2z)k = 0,$$

so that we need to solve the system of linear equations

$$\begin{array}{rcrcrcl} 6x & - & 3y & + & 2z & = & 0 \\ & & 3y & - & 3z & = & 0 \\ & & & & z & = & 6. \end{array}$$

The unique solution to this system is $x = a_{1,3} = 1$, $y = a_{2,3} = 6$, and $z = a_{3,3} = 6$, so that $k^3 = C(k,1) + 6C(k,2) + 6C(k,3)$. Therefore,

$$\begin{aligned} \sum_{k=1}^{n} k^3 &= \sum_{k=1}^{n} (C(k,1) + 6C(k,2) + 6C(k,3)) \\ &= \sum_{k=1}^{n} C(k,1) + 6\sum_{k=1}^{n} C(k,2) + 6\sum_{k=1}^{n} C(k,3) \end{aligned}$$

$$
\begin{aligned}
&= C(n+1,2) + 6C(n+1,3) + 6C(n+1,4) \\
&\quad \text{by Chu's Theorem (Theorem 1.5.1)} \\
&= \frac{n(n+1)}{2} + 6\frac{(n-1)n(n+1)}{6} + 6\frac{(n-2)(n-1)n(n+1)}{24} \\
&= \frac{n(n+1)}{2}\left(1 + 2(n-1) + \frac{n^2-3n+2}{2}\right) \\
&= \frac{n(n+1)}{2}\left(\frac{4n-2+(n^2-3n+2)}{2}\right) = \left(\frac{n(n+1)}{2}\right)\left(\frac{n^2+n}{2}\right) \\
&= \left(\frac{n(n+1)}{2}\right)\left(\frac{n(n+1)}{2}\right) = \left(\frac{n(n+1)}{2}\right)^2,
\end{aligned}
$$

as expected.

Definition 1.5.3. Let C_n be the $n \times n$ *Pascal matrix* whose (i,j)-entry is binomial coefficient $C(i,j)$ for $1 \leq i,j \leq n$.

Theorem 1.5.4. Alternating-Sign Theorem.
The Pascal matrix C_n is invertible; the (i,j)-entry of C_n^{-1} is $(-1)^{i+j}C(i,j)$.

Theorem 1.5.5. If m and r are positive integers, the coefficient of $C(k,r)$ in the equation $k^m = \sum_{r=1}^m a_{r,m}C(k,r)$ is given by

$$a_{r,m} \sum_{t=1}^{m} (-1)^{r+t} C(r,t) t^m.$$

Lemma 1.5.8. If $n > 0$, then $\sum_{r=0}^n (-1)^r C(n,r) = 0$.