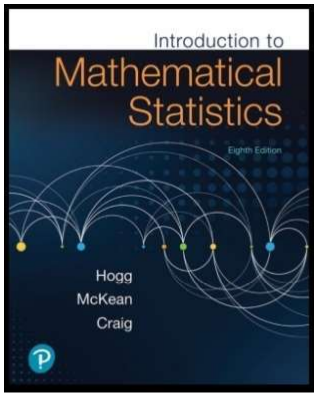
Mathematical Statistics 1

Chapter 1. Introduction to Probability

1.2. Sets—Proofs of Theorems



Theorem 1.2.A

Theorem 1.2.A. For any sets (events) A, B, and C we have

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C) \text{ and } A \cup (C \cap C) = (A \cup B) \cap (A \cup C).$$

These are the *distributive laws*.

Proof. We establish each claim by showing the set on the left side is a subset of the set on the right side and conversely.

Let $x \in A \cap (C \cup C)$. Then by Definition 1.2.4, both $x \in A$ and $x \in B \cup C$. By Definition 1.2.3, $x \in B \cap C$ implies that either $x \in B$ or $x \in C$. If $x \in B$ then we have $x \in A \cap B$ and if $x \in C$ then we have $x \in A \cap C$ (by Definition 1.2.4). So we have that either $x \in A \cap B$ or $x \in A \cap C$. By Definition 1.2.3, this implies $x \in (A \cap B) \cup (A \cap C)$. Since x is an arbitrary element of $A \cap (B \cup C)$ then we have $A \cap (B \cup C) \subset (A \cap B) \cup (A \cap C)$ by Definition 1.2.2.

Theorem 1.2.A (continued 1)

Theorem 1.2.A. For any sets (events) A, B, and C we have

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C) \text{ and } A \cup (C \cap C) = (A \cup B) \cap (A \cup C).$$

These are the *distributive laws*.

Proof (continued). Now let $y \in (A \cap B) \cup (A \cap C)$. Then by Definition 1.2.3, either $y \in A \cap B$ or $y \in A \cap C$. If $y \in A \cap B$ then by Definition 1.2.4, both $y \in A$ and $y \in B$. Therefore both $y \in A$ and $y \in B \cup C$ (since $y \in B$; by Definition 1.2.3). So, by Definition 1.2.3, $y \in A \cap (B \cup C)$. If $y \in A \cap C$ then by Definition 1.2.4, both $y \in A$ and $y \in C$. Therefore both $y \in A$ and $y \in B \cup C$ (since $y \in C$; by Definition 1.2.3). So by Definition 1.2.3, $y \in A \cap (B \cup C)$. Since y is an arbitrary element of $A \cap B) \cup (A \cap C)$ then $(A \cap B) \cup (A \cap C) \subset A \cap (B \cup C)$ by Definition 1.2.2. Therefore, by the definition of "equal" (also Definition 1.2.2), $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$, as claimed.

Theorem 1.2.A (continued 2)

Theorem 1.2.A. For any sets (events) A, B, and C we have

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C) \text{ and } A \cup (B \cap C) = (A \cup B) \cap (A \cup C).$$

Proof (continued). We now establish the second claim, but this time we go a little faster and reference the definitions less. Let $x \in A \cup (B \cap C)$. Then $x \in A$ or $x \in B \cap C$. If $x \in A$ then $x \in A \cup B$ and $x \in A \cup C$, and hence $x \in (A \cup B) \cap (A \cup C)$. If $x \in B \cap C$ then $x \in B$ and $x \in C$. So $x \in A \cup B$ and $x \in A \cup C$, so that $x \in (A \cup B) \cap (A \cup C)$. Since x is an arbitrary element of $A \cup (B \cap C)$ then $A \cup (B \cap C) \subset (A \cup B) \cap (A \cap C)$. Let $y \in (A \cup B) \cap (A \cup C)$. Then both $y \in A \cup B$ and $y \in A \cup C$; that is, BOTH $y \in A$ or $y \in B$ AND $y \in A$ or $y \in C$. If $y \in A$ then $y \in A \cup (B \cap C)$. If $y \notin A$ then we must have both $y \in B$ and $y \in C$, that is $y \in B \cap C$, and hence $y \in B \cap C$. So if $y \notin A$ then $y \in A \cup (B \cap C)$. Since y is an arbitrary element of $(A \cup B) \cap (A \cup C)$ then $(A \cup B) \cap (A \cup C) \subset A \cup (B \cap C)$. Therefore, $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$, as claimed. □

Theorem 1.2.B

Theorem 1.2.B. De Morgan's Laws. For any two sets (events) A and B, we have

$$(A \cap B)^c = A^c \cup B^c \text{ and } (A \cup B)^c = A^c \cap B^c.$$

Proof (Exercise 1.2.4(b). We establish each claim by showing the set on the left side is a subset of the set on the right side and conversely.

Let $x \in (A \cap B)^c$. Then $x \notin A \cap B$. Since $A \cap B$ consists of elements in both A and B by Definition 1.2.4, so x is not in both A and B; that is, either x is not in A or x is not in B. Hence either $x \in A^c$ or $x \in B^c$; that is, $x \in A^c \cup B^c$. Since x is an arbitrary element of $(A \cap B)^c$, then $(A \cap B)^c \subset A^c \cup B^c$. Let $y \in A^c \cup B^c$. Then either $y \in A^c$ or $y \in B^c$ by Definition 1.2.3. Since y is an arbitrary element of $A^c \cap B^c$, then $A^c \cup B^c \subset (A \cap B)^c$. Therefore, by the definition of "equal" (Definition 1.2.2), $(A \cap B)^c = A^c \cup B^c$, as claimed.

Theorem 1.2.B (continued)

Theorem 1.2.B. De Morgan's Laws. For any two sets (events) A and B, we have

$$(A \cap B)^c = A^c \cup B^c \text{ and } (A \cup B)^c = A^c \cap B^c.$$

Proof (continued). Let $x \in (A \cup B)^c$. Then $x \notin A \cup B$. Since $A \cup B$ consists of all elements in either A or B by Definition 1.2.3, then x is in neither A nor B. That is, both $x \in A^c$ and $x \in B^c$. So, by Definition 1.2.4, $x \in A^c \cap B^c$. Since x is an arbitrary element of $(A \cup B)^c$ then $(A \cup B)^c \subset A^c \cap B^c$. Now let $y \in A^c \cap B^c$. Then by Definition 1.2.4 both $y \in A^c$ and $y \in B^c$. So both $y \notin A$ and $y \notin B$. Hence $y \notin A \cup B$ by Definition 1.2.3. That is, $y \in (A \cup B)^c$. Since y is an arbitrary element of $A^c \cap B^c$ then $A^c \cap B^c \subset (A \cup B)^c$. Therefore, by the definition of "equal" (Definition 1.2.2), $(A \cup B)^c = A^c \cap B^c$, as claimed. □