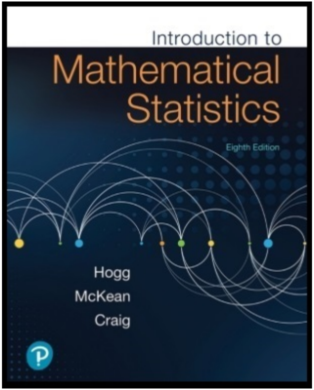
Mathematical Statistics 1

Chapter 1. Introduction to Probability

1.2. Sets—Proofs of Theorems



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Theorem 1.2.A

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$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ and $A \cup (C \cap C) = (A \cup B) \cap (A \cup C)$.

These are the *distributive laws*.

Proof. We establish each claim by showing the set on the left side is a subset of the set on the right side and conversely.

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Let $x \in A \cap (C \cup C)$. Then by Definition 1.2.4, both $x \in A$ and $x \in B \cup C$. By Definition 1.2.3, $x \in B \cap C$ implies that either $x \in B$ or $x \in C$.

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$$(A \cap B)^c = A^c \cup B^c \text{ and } (A \cup B)^c = A^c \cap B^c.$$

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