Chapter 1. Introduction to Probability

1.2. Sets—Proofs of Theorems
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Theorem 1.2.A

Theorem 1.2.A. For any sets (events) $A$, $B$, and $C$ we have

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C) \text{ and } A \cup (C \cap C) = (A \cup B) \cap (A \cup C).$$

These are the distributive laws.

Proof. We establish each claim by showing the set on the left side is a subset of the set on the right side and conversely.
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Let $x \in A \cap (C \cup C)$. Then by Definition 1.2.4, both $x \in A$ and $x \in B \cup C$. By Definition 1.2.3, $x \in B \cap C$ implies that either $x \in B$ or $x \in C$. 
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\textbf{Proof (continued).} Now let $y \in (A \cap B) \cup (A \cap C)$. Then by Definition 1.2.3, either $y \in A \cap B$ or $y \in A \cap C$. If $y \in A \cap B$ then by Definition 1.2.4, both $y \in A$ and $y \in B$. Therefore both $y \in A$ and $y \in B \cup C$ (since $y \in B$; by Definition 1.2.3). So, by Definition 1.2.3, $y \in A \cap (B \cup C)$. 


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A \cap (B \cup C) = (A \cap B) \cup (A \cap C) \quad \text{and} \quad A \cup (C \cap C) = (A \cup B) \cap (A \cup C).
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**Proof (continued).** We now establish the second claim, but this time we go a little faster and reference the definitions less. Let $x \in A \cup (B \cap C)$. Then $x \in A$ or $x \in B \cap C$. If $x \in A$ then $x \in A \cup B$ and $x \in A \cup C$, and hence $x \in (A \cup B) \cap (A \cup C)$. If $x \in B \cap C$ then $x \in B$ and $x \in C$. So $x \in A \cup B$ and $x \in A \cup C$, so that $x \in (A \cup B) \cap (A \cup C)$. Since $x$ is an arbitrary element of $A \cup (B \cap C)$ then $A \cup (B \cap C) \subseteq (A \cup B) \cap (A \cup C)$. Let $y \in (A \cup B) \cap (A \cup C)$. Then both $y \in A \cup B$ and $y \in A \cup C$; that is, BOTH $y \in A$ or $y \in B$ AND $y \in A$ or $y \in C$. If $y \in A$ then $y \in A \cup (B \cap C)$. If $y \notin A$ then we must have both $y \in B$ and $y \in C$, that is $y \in B \cap C$, and hence $y \in B \cap C$. So if $y \notin A$ then $y \in A \cup (B \cap C)$. Since $y$ is an arbitrary element of $(A \cup B) \cap (A \cup C)$ then $(A \cup B) \cap (A \cup C) \subseteq A \cup (B \cap C)$. Therefore, $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$, as claimed.
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Theorem 1.2.B. De Morgan’s Laws. For any two sets (events) $A$ and $B$, we have

$$(A \cap B)^c = A^c \cup B^c \quad \text{and} \quad (A \cup B)^c = A^c \cap B^c.$$ 

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Let \( y \in A^c \cup B^c \). Then either \( y \in A^c \) or \( y \in B^c \) by Definition 1.2.3. Since \( y \) is an arbitrary element of \( A^c \cap B^c \), then \( A^c \cup B^c \subset (A \cap B)^c \). Therefore, by the definition of “equal” (Definition 1.2.2), \((A \cap B)^c = A^c \cup B^c\), as claimed.
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Theorem 1.2.B (continued)

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