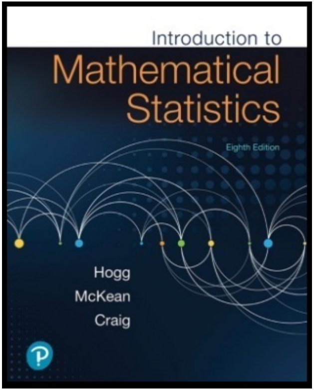
Mathematical Statistics 1

Chapter 1. Introduction to Probability

1.3. The Probability Set Function—Proofs of Theorems



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$$1 = P(\mathcal{C}) = P(A \cup A^c) = P(A) + P(A^c),$$

so that $P(A) = 1 - P(A^c)$, as claimed. □

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Theorem 1.3.3. If A and B are events such that $A \subset B$, then $P(A) \leq P(B)$ (in measure theory, this is called *monotonicity*).

Proof. We have $B = A \cup (A^c \cap B)$, so by Definition 1.3.1(3) (countable additivity) $P(B) = P(A) + P(A^c \cap B)$ and by Definition 1.3.1(1), $P(A^c \cap B) \geq 0$ so that

$$P(B) = P(A) + P(A^c \cap B) \geq P(A),$$

or $P(A) \leq P(B)$, as claimed. $\square$

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Theorem 1.3.4

Theorem 1.3.4. For each event $A \in \mathcal{B}$ we have $0 \leq P(A) \leq 1$.

Proof. Since $\varnothing \subset A \subset \mathcal{C}$ then by Theorem 1.3.2, Theorem 1.3.3, and Definition 1.3.1(2),

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Theorem 1.3.5. If A and B are events in $\mathcal{C}$, then
$P(A \cup B) = P(A) + P(B) - P(A \cap B)$.

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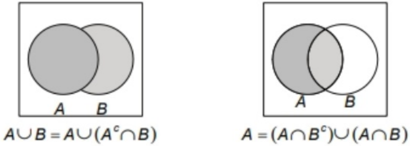

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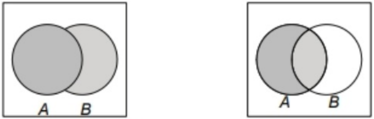

$A \cup B = A \cup (A^c \cap B)$ $\quad$ $A = (A \cap B^c) \cup (A \cap B)$

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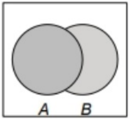

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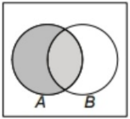

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Exercise 1.3.4

Exercise 1.3.4. If the sample space is $\mathcal{C} = C_1 \cup C_2$ and if $P(C_1) = 0.8$ and $P(C_2) = 0.5$, find $P(C_1 \cap C_2)$.

Solution. With $A = C_1$ and $B = C_2$, we have from Theorem 1.3.4 that

$$P(\mathcal{C} = P(C_1 \cup C_2) = P(C_1) + P(C_2) - P(C_1 \cap C_2)$$

or $1 = (0.8) + (0.5) - P(C_1 \cap C_2)$ or $\boxed{P(C_1 \cap C_2) = 0.3}$. □

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Exercise 1.3.6

Exercise 1.3.6. If the sample space is $\mathcal{C} = \{c \mid -\infty < c < \infty\}$ and if $C \subset \mathcal{C}$ is a set for which the integral $\int_C e^{-|x|}\,dx$ exists, show that this set function is not a probability set function. What constant do we multiply the integral by to make it a probability function?

Solution. With $C = \mathcal{C} = \mathbb{R}$ we have

$$\begin{aligned}
\int_{\mathbb{R}} e^{-|x|}\,dx &= \int_{-\infty}^{\infty} e^{-|x|}\,dx \\
&= 2\int_0^{\infty} e^{-|x|}\,dx \text{ since } e^{-|x|} \text{ is an even function} \\
&= 2\int_0^{\infty} e^{-x}\,dx \text{ since } x \geq 0 \text{ here} \\
&= 2\lim_{b\to\infty}\left(\int_0^b e^{-x}\,dx\right) = 2\lim_{b\to\infty}\left(-e^{-x}\Big|_0^b\right) \\
&= 2\lim_{b\to\infty}(-e^{-b}+1) = 2(0+1) = 2.
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Exercise 1.3.6 (continued 1)

Solution (continued). So $\int_C e^{-|x|}\,dx$ is not a probability set function because applying it to $C = \mathcal{C}$ does not yield a probability of a (in violation of Definition 1.3.1(2)). If we define $P(C) = \frac{1}{2}\int_C e^{-|x|}\,dx$ then we have $P(|calC) = 1$ and Definition 1.3.1(2) is then satisfied. We should feel comfortable with the claim that $P(\varnothing) = \int_\varnothing e^{-|x|}\,dx = 0$ (though this is never technically defined for Riemann integrals), so that Definition 1.3.1(1) is satisfied.

But justifying Definition 1.3.1(3), countable additivity, is more complicated. If the integral is a Riemann integral then there are a lot of restrictions on the collection $\mathcal{B}$ of events. If the integral is a Lebesgue integral then the collection of events $\mathcal{B}$ is the σ-field (or σ-algebra) of Lebesgue measurable sets, which includes lots of sets of real numbers (probably every subset of $\mathbb{R}$ you can think of... certainly every subset that *I* can think of... almost...).

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But justifying Definition 1.3.1(3), countable additivity, is more complicated. If the integral is a Riemann integral then there are a lot of restrictions on the collection $\mathcal{B}$ of events. If the integral is a Lebesgue integral then the collection of events $\mathcal{B}$ is the σ-field (or σ-algebra) of Lebesgue measurable sets, which includes lots of sets of real numbers (probably every subset of $\mathbb{R}$ you can think of... certainly every subset that *I* can think of... almost...).

Exercise 1.3.6 (continued 2)

Exercise 1.3.6. If the sample space is $\mathcal{C} = \{c \mid -\infty < c < \infty\}$ and if $C \subset \mathcal{C}$ is a set for which the integral $\int_C e^{-|x|}\,dx$ exists, show that this set function is not a probability set function. What constant do we multiply the integral by to make it a probability function?

Solution (continued). One of the properties of the Lebesgue integral is countable additivity:

$$\int_{\cup_{n=1}^{\infty} A_n} e^{-|x|} = \sum_{n=1}^{\infty} \left(\int_{A_n} e^{-|x|} \right)$$

so that $P\left(\cup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} P(A_n)$, and Definition 1.3.1(3) is satisfied. For more details on properties of Lebesgue integrals, see my online notes for Real Analysis 1. □

Exercise 1.3.9

Exercise 1.3.9. Determine the probability of being dealt a full house, i.e., three-of-a-kind and two-of-a-kind.

Solution. The suit of the three-of-a-kind can be chosen in $\binom{13}{1} = 13$ ways and the suit of the two-of-a-kind can then be chose in $\binom{12}{1} = 12$ ways. The three cards in the three-of-a-kind can then be chosen in $\binom{4}{3}$ ways and the two cards in the two-of-a-kind can then be chosen in $\binom{4}{2}$ ways. So the probability of being dealt a full house is

$$\frac{\binom{13}{1}\binom{12}{1}\binom{4}{3}\binom{4}{2}}{\binom{52}{5}} = \frac{(13)(12)(4)(6)}{2{,}598{,}960} \approx 0.00144.$$

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Theorem 1.3.6

Theorem 1.3.6. Continuity of the Probability Functions.
Let $\{C_n\}$ be a nondecreasing sequence of events. Then

$$\lim_{n\to\infty} P(C_n) = P\left(\lim_{n\to\infty} C_n\right) = P\left(\cup_{n=1}^{\infty} C_n\right).$$

Let $\{C_n\}$ be a nonincreasing sequence of sets. Then

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Define $R_1 = C_1$ and $R_n = C_n \cap C_{n-1}^c$, for $n \geq 2$. Notice that since the events are in a σ-field then R_n is also an event. Then $R_m \cap R_n = \varnothing$ for $m \neq n$ (since, with $m < n$ say, $R_m \subset C_m$ but $R_n \subset C_{n-1}^c$ and since the sequence is nondecreasing then $C_m \subset C_{n-1}$, here $m \leq n-1$, and so $C_m \cap C_{n-1}^c = \varnothing$).

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Theorem 1.3.6 (continued 1)

Proof (continued). Also, $\dot\cup_{n=1}^{\infty} R_n = \cup_{n=1}^{\infty} C_n$ since $R_n \subset C_n$ for $n \geq 1$ and any $x \in \cup_{n=1}^{\infty} C_n$ is in some C_N for a smallest value of $N \in \mathbb{N}$ so that $x \in R_N = C_N \cap C_{N-1}^c$ (since N is the smallest such value then $x \not\in X_{N-1}$ and so $x \in C_{N-1}^c$; we need $X_0 = \varnothing$ here). Since $R_n = C_n \cap C_{n-1}^c$ then

$$R_n \dot\cup C_{n-1}^c = (C_n \cap C_{n-1}^c) \dot\cup C_{n-1}$$

or $R_n \dot\cup C_{n-1} = C_n$ and so by Definition 1.3.1(3), countable additivity,

$$P(R_n \dot\cup C_{n-1}) = P(R_n) + P(C_{n-1}) = P(C_n)$$

or $P(R_n) = P(C_n) - P(C_{n-1})$. So for any $N \in \mathbb{N}$ we have

$$\sum_{n=1}^{N} P(R_n) = \sum_{n=1}^{N} (P(C_n) - P(C_{n-1})) = P(C_N) - P(C_0) = P(C_N).$$

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Theorem 1.3.6 (continued 2)

Proof (continued). ...

$$\sum_{n=1}^{N} P(R_n) = \sum_{n=1}^{N} (P(C_n) - P(C_{n-1})) = P(C_N) - P(C_0) = P(C_N).$$

So

$$\begin{aligned} P\left(\lim_{n\to\infty} C_n\right) &= P\left(\cup_{n=1}^{\infty} C_n\right) = P\left(\dot\cup_{n=1}^{\infty} R_n\right) \\ &= \sum_{n=1}^{\infty} P(R_n) \text{ by Definition 1.3.1(3), countable additivity} \\ &= \lim_{N\to\infty}\left(\sum_{n=1}^{N} P(R_n)\right) = \lim_{N\to\infty} P(C_N) = \lim_{n\to\infty} P(C_n), \end{aligned}$$

as claimed. □

Theorem 1.3.7

Theorem 1.3.7. Boole's Inequality/Countable Subadditivity.
Let $\{C_n\}$ be an arbitrary sequence of events. Then

$$P\left(\cup_{n=1}^{\infty} C_n\right) \leq \sum_{n=1}^{\infty} P(C_n).$$

Proof. Define $D_n = \cup_{i=1}^{n} C_i$. Then $\{D_n\}$ is an increasing sequence of events that converge to $\cup_{n=1}^{\infty} C_n$. Also $D_j = D_{j-1} \cup C_j$ for all $j \geq 2$. So by Theorem 1.3.5,

$$P(D_j) = P(D_{j-1} \cup C_j) \leq P(D_{j-1}) + P(C_j),$$

or $P(D_j) - P(D_{j-1}) \leq P(C_j)$.

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$$P(D_j) = P(D_{j-1} \cup C_j) \leq P(D_{j-1}) + P(C_j),$$

or $P(D_j) - P(D_{j-1}) \leq P(C_j)$.

Theorem 1.3.7 (continued)

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$$P\left(\cup_{n=1}^{\infty} C_n\right) \leq \sum_{n=1}^{\infty} P(C_n).$$

Proof (continued). So by Theorem 1.3.1,

$$P\left(\cup_{i=1}^{\infty} C_n\right) = \left(\cup_{i=1}^{\infty} D_c\right) = \lim_{n\to\infty} P(D_n)$$

$$= \lim_{n\to\infty} \left(P(D_1) + \sum_{j=2}^{n} (P(D_j) - P(D_{j-1})) \right) \leq \lim_{n\to\infty} \left(P(C_1) + \sum_{j=2}^{\infty} P(C_j) \right)$$

$$= \sum_{n=1}^{\infty} P(C_n),$$

as claimed. □