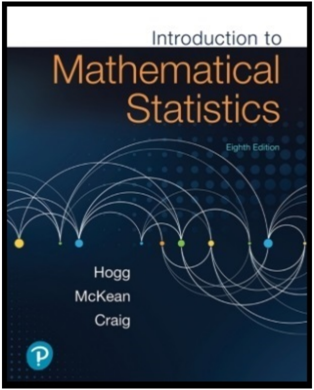
Mathematical Statistics 1

Chapter 1. Introduction to Probability

1.4. Conditional Probability and Independence—Proofs of Theorems



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Theorem 1.4.A

Theorem 1.4.A. Let $A, B, B_1, B_2, \ldots$ be events with $P(A) > 0$. Then

1. $P(B \mid A) \geq 0$.
2. $P(A \mid A) = 1$.
3. $P(\cup_{n=1}^{\infty} B_n \mid A) = \sum_{n=1}^{\infty} P(B_n \mid A)$ provided $B_1, B_2, \ldots$ are mutually exclusive.

Proof. (1) Since $P(A) > 0$, $P(A \cap B) \geq 0$, and $P(B \mid A) = \dfrac{P(A \cap B)}{P(A)}$, then $P(A \mid B) \geq 0$. □

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(2) Since $P(A) > 0$ and by Definition 1.4.1,

$$P(A \mid B) = \frac{P(A \cap A)}{P(A)} = \frac{P(A)}{P(A)} = 1.$$

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Exercise 1.4.6

Exercise 1.4.6. A drawer contains eight different pairs of socks. If six socks are taken at random and without replacement, compute the probability that there is at least one matching pair among these six socks.

Solution. We compute the probability of the complement event that there is no pair of socks. We put no condition on the first sock, so that probability that the first sock does not form a pair(!) is $16/16 = 1$. Next, we compute the probability that the second sock does not form a pair with the first sock. The probability of this is $14/15$. The probability that the third sock does not form a pair with either the first sock or the second sock GIVEN that the first two socks do not form a pair is $12/14$.

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Exercise 1.4.6 (continued)

Exercise 1.4.6. A drawer contains eight different pairs of socks. If six socks are taken at random and without replacement, compute the probability that there is at least one matching pair among these six socks.

Solution (continued). By the multiplication rule, the probability that no socks form a pair is

$$\left(\frac{16}{16}\right)\left(\frac{14}{15}\right)\left(\frac{12}{14}\right)\left(\frac{10}{13}\right)\left(\frac{8}{12}\right)\left(\frac{6}{11}\right) = \frac{32}{143}.$$

So the probability that there is at least one pair is
$1 - \frac{32}{143} = \boxed{\frac{111}{143} \approx 0.776}$. □

Theorem 1.4.B

Theorem 1.4.B. Law of Total Probability.
Let $A_1, A_2, \ldots, A_k$ be events such that $P(A_i) > 0$ for $i = 1, 2, \ldots, k$ and are mutually exclusive and exhaustive (that is, $\mathcal{C} = \cup_{i=1}^k A_k$). Let B be another event such that $P(B) > 0$. Then

$$P(B) = \sum_{i=1}^{k} P(A_i) P(B \mid A_i).$$

Proof. We have $B = B \cap \mathcal{C} = B \cap \left(\cup_{i=1}^k A_i\right) = \cup_{i=1}^k (B \cap A_i)$. So

$$P(B) = P\left(\cup_{i=1}^k (B \cap A_i)\right) = \sum_{i=1}^{k} P(B \cap A_i) = \sum_{i=1}^{k} P(A_i) P(B \mid A_i),$$

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Theorem 1.4.1. Bayes' Theorem.
Let $A_1, A_2, \ldots, A_k$ be events such that $P(A_i) > 0$ for $i = 1, 2, \ldots, k$. Assume that $A_1, A_2, \ldots, A_k$ form a partition of the sample space $\mathcal{C}$. Let B be any event. Then for each $j = 1, 2, \ldots, k$ we have

$$P(A_j \mid B) = \frac{P(A_j)P(B \mid A_j)}{\sum_{i=1}^{k} P(A_i)P(B \mid A_i)}.$$

Proof. By Definition 1.4.1, for each j we have

$$\begin{aligned} P(A_j \mid B) &= \frac{P(B \cap A_j)}{P(B)} = \frac{P(A_j)P(B \mid A_j)}{P(B)} \\ &= \frac{P(A_j)P(B \mid A_j)}{\sum_{i=1}^{k} P(B \mid A_i)} \text{ by the Law of Total Probability,} \end{aligned}$$

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Example 1.4.5

Example 1.4.5. Suppose Bowl A_1 contains three red chips and seven blue chips, and Bowl A_2 contains eight red chips and two blue chips. A 6-sided die is cast and Bowl A_1 is selected if five or six spots show on the side that is up; otherwise Bowl A_2 is selected. Therefore $P(A_1) = 2/6 = 1/3$ and $P(A_2) = 4/6 = 2/3$. A chip is removed from the selected bowl. Let B denote the event that the selected chip is red. Then $P(B \mid A_1) = 3/10$ and $P(B \mid A_2) = 8/10 = 4/5$. We can calculate the conditional probabilities $P(A_1 \mid B)$ and $P(A_2 \mid B)$ using Bayes' Theorem:

$$P(A_1 \mid B) = \frac{P(A_1)P(B \mid A_1)}{P(A_1)P(B \mid A_1) + P(A_2)P(B \mid A_2)}$$

$$= \frac{(1/3)(3/10)}{(1/3)(3/10) + (2/3)(4/5)} = \frac{3}{19}$$

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Example 1.4.5 (continued)

Solution (continued). . . . and

$$P(A_2 \mid B) = \frac{P(A_2)P(B \mid A_2)}{P(A_1)P(B \mid A_1) + P(A_2)P(B \mid A_2)}$$

$$= \frac{(2/3)(4/5)}{(1/3)(3/10) + (2/3)(4/5)} = \frac{16}{19}.$$

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Theorem 1.4.C

Theorem 1.4.C. Suppose A and B are independent events. The the following three pairs of events are independent: A^c and B, A and B^c, and A^c and B^c.

Proof (this includes Exercise 1.4.11). Since $B = (A^c \cap B) \cup (A \cap B)$ then $P(B) = P(A^c \cap B) + P(A \cap B)$ and so

$$\begin{aligned} P(A^c \cap B) &= P(B) - P(A \cap B) \\ &= P(B) - P(A)P(B) \text{ since } A \text{ and } B \text{ are independent} \\ &= (1 - P(A))P(B) = P(A^c)P(B), \end{aligned}$$

so A^c and B are independent, as claimed.

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Proof (continued). Since $A^c = (A^c \cap B^c) \mathbin{\dot\cup} (A^c \cap B)$ then $P(A^c) = P(A^c \cap B^c)P(A^c \cap B)$ and so

$$\begin{aligned} P(A^c \cap B^c) &= P(A^c) - P(A^c \cap B) \\ &= P(A^c) - P(A^c)P(B) \text{ since } A^c \text{ and } B \\ &\quad\ \text{are independent, as shown above} \\ &= P(A^c)(a - P(B)) = P(A^c)P(B^c), \end{aligned}$$

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Exercise 1.4.18

Exercise 1.4.18. A die cast independently until the first 6 appears. If the casting stops on an odd number of times then Bob wins, otherwise Joe wins. **(a)** Assuming the die is fair, what is the probability that Bob wins? **(b)** Let p denote the probability of a 6. Show that the game favors Bob for all p with $0 < p < 1$.

Solution. **(a)** Since the die is fair, we expect the probability of casting a 6 is $1/6$ and so the probability of casting a non-6 is $5/6$. Let $n \in \mathbb{N}$. Then $2n-1$ is odd and the probability that the casting ends on $2n-1$ casts is $(5/6)^{2n-2}(1/6)$. Also, $2n$ is even and the probability the casting ends on $2n$ casts is $(5/6)^{2n-1}(1/6)$.

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$$\sum_{n=1}^{\infty}\left(\frac{5}{6}\right)^{2n-1}\left(\frac{1}{6}\right) = \left(\frac{5}{6}\right)^{-1}\left(\frac{1}{6}\right)\sum_{n=1}^{\infty}\left(\frac{25}{36}\right)^{n}$$

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Exercise 1.4.18 (continued 1)

Solution (continued). ...So the probability that Bob wins is

$$\sum_{n=1}^{\infty}\left(\frac{5}{6}\right)^{2n-1}\left(\frac{1}{6}\right) = \frac{36}{150}\frac{25/36}{1-(25/26)} \text{ since we have}$$

a geometric series with ratio $r = 25/36$

$$= \frac{1}{6}\frac{1}{11/36} = \boxed{\frac{6}{11}}.$$

(b) With the probability of casting a 6 as p, then the probability of casting a non-6 is $1-p$ so, as above, the probability that Bob wins is

$$\sum_{n=1}^{\infty}(1-p)^{2n-2}p = \frac{p}{(1-p)^2}\sum_{n=1}^{\infty}((1-p)^2)^n$$

$$= \frac{p}{(1-p)^2}\frac{(1-p)^2}{1-(1-p)^2} = \frac{p}{2p-p^2} = \boxed{\frac{1}{2-p}}.$$

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Exercise 1.4.18 (continued 2)

Exercise 1.4.18. A die cast independently until the first 6 appears. If the casting stops on an odd number of times then Bob wins, otherwise Joe wins. **(a)** Assuming the die is fair, what is the probability that Bob wins? **(b)** Let p denote the probability of a 6. Show that the game favors Bob for all p with $0 < p < 1$.

Solution (continued). As above, the probability that Joe wins is

$$\sum_{n=1}^{\infty}(1-p)^{2n-1}p = \frac{p}{1-p}\sum_{n=1}^{\infty}(1-p)^{2n}$$

$$= \frac{p}{1-p}\frac{(1-p)^2}{1-(1-p)^2} = \frac{p(1-p)}{2p-p^2} = \boxed{\frac{1-p}{2-p}}.$$

For $0 < 0 < 1$, $1-p < 1$ and $\dfrac{1-p}{2-p} < \dfrac{1}{2-p}$ so that the probability that Bob wins is greater than the probability that Joe wins. □