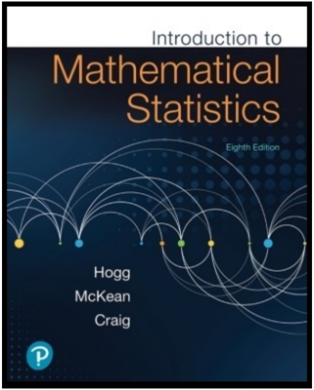
Mathematical Statistics 1

Chapter 1. Introduction to Probability

1.5. Random Variables—Proofs of Theorems



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Theorem 1.5.1

Theorem 1.5.1. Let X be a random variable with cumulative distribution function $F(x)$. Then

(a) For all a and b, if $a < b$ then $F(a) \le F(b)$ (i.e., F is nondecreasing).

(b) $\lim_{x \to -\infty} F(x) = 0$.

(c) $\lim_{x \to \infty} F(x) = 1$.

(d) $\lim_{x \downarrow x_0} F(x) = \lim_{x \to x_0^+} F(x) = F(x_0)$ (i.e., F is right continuous).

Proof. (a) With $a < b$ we have

$$\{X \le a\} = \{x \in \mathcal{C} \mid X(c) \le a\} \subset \{c \in \mathcal{C} \mid X(c) \le b\} = \{X \le b\},$$

so by monotonicity of P, Theorem 1.3.3, we have $P(X \le a) \le P(X \le b)$ or $F(A) = P(X \le a) \le P(X \le b) = F(b)$, as claimed. □

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Theorem 1.5.1 (continued 1)

Proof (continued). (b) (This is part of Exercise 1.5.10.) For $n \in \mathbb{N}$, define $x_n = -n$ and define $C_n = \{c \in \mathcal{C} \mid X(c) \leq -n\} = \{X \leq -n\}$. So $x_n \to -\infty$, the sequence of sets $\{C_n\}$ is nonincreasing and $\lim_{n\to\infty} C_n = \cap_{n=1}^{\infty} C_n = \varnothing$. So by the Continuity of the Probability Function, Theorem 1.3.6, we have

$$\lim_{n\to\infty} P(X \leq x_n) = \lim_{n\to\infty} P(C_n) = P\left(\lim_{n\to\infty} C_n\right) = P(\varnothing) = 0,$$

where $P(\varnothing) = 0$ by Theorem 1.3.2. We now need to replace the limit of the sequence with the limit of the probability function.

Let $\varnothing > 0$. Then there exists $N \in \mathbb{N}$ such that for all $n \geq N$ we have $|P(X \leq x_n) - 0| < \varepsilon$ or $0 \leq P(X \leq x_n) < \varepsilon$. Let $x \in \mathbb{R}$ where $x \leq -N$. By part (a), $0 \leq F(x) \leq F(-N) = P(X \leq x_n) < \varepsilon$. So for each $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that if $x \leq -N$ then $|F(x) - 0| < \varepsilon$. So by the definition of $\lim_{x\to-\infty} F(x) = L$, we have $\lim_{x\to-\infty} F(x) = 0$, as claimed. □

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Theorem 1.5.1 (continued 2)

Proof (continued). (c) (This is part of Exercise 1.5.10.) For $n \in \mathbb{N}$, define $x_n = n$ and define $C_n = \{c \in \mathcal{C} \mid X(c) \leq n\} = \{X \leq n\}$. So $x_n \to \infty$, the sequence of sets $\{C_n\}$ is nondecreasing and $\lim_{n\to\infty} C_n = \cup_{n=1}^{\infty} C_n = \mathbb{R}$. so by the Continuity of the Probability Function, Theorem 1.3.5, we have

$$\lim_{n\to\infty} P(X \leq x_n) = \lim_{n\to\infty} P(C_n) = P\left(\lim_{n\to\infty} C_n\right) = P(\mathbb{R}) = 1,$$

where $P(\mathbb{R}) = 1$ by Definition 1.3.1(2). We now need to replace the limit of the sequence with the limit of the probability function.

Let $\varepsilon > 0$. Then there exists $N \in \mathbb{N}$ such that for all $n \geq N$ we have $|P(X \leq x_n) = 1| < \varepsilon$, or $1 - \varepsilon < P(X \leq x_n) \leq 1$, or $1 - \varepsilon < F(x_n) \leq 1$. Let $x \in \mathbb{R}$ where $x \geq N$. By part (a), $F(x) \geq F(N) = F(x_n)$ or $1 - \varepsilon < F(x_n) \leq F(x) \leq 1$, or $|F(x) - F(x_n)| < \varepsilon$. So for each $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that if $x \geq N$ then $|F(x) - F(x_n)| < \varepsilon$. So by the definition of $\lim_{x\to\infty} F(x) = L$ we have $\lim_{x\to\infty} F(x) = 1$, as claimed. □

Theorem 1.5.1 (continued 2)

Proof (continued). (c) (This is part of Exercise 1.5.10.) For $n \in \mathbb{N}$, define $x_n = n$ and define $C_n = \{c \in \mathcal{C} \mid X(c) \leq n\} = \{X \leq n\}$. So $x_n \to \infty$, the sequence of sets $\{C_n\}$ is nondecreasing and $\lim_{n\to\infty} C_n = \cup_{n=1}^{\infty} C_n = \mathbb{R}$. so by the Continuity of the Probability Function, Theorem 1.3.5, we have

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Let $\varepsilon > 0$. Then there exists $N \in \mathbb{N}$ such that for all $n \geq N$ we have $|P(X \leq x_n) = 1| < \varepsilon$, or $1 - \varepsilon < P(X \leq x_n) \leq 1$, or $1 - \varepsilon < F(x_n) \leq 1$. Let $x \in \mathbb{R}$ where $x \geq N$. By part (a), $F(x) \geq F(N) = F(x_n)$ or $1 - \varepsilon < F(x_n) \leq F(x) \leq 1$, or $|F(x) - F(x_n)| < \varepsilon$. So for each $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that if $x \geq N$ then $|F(x) - F(x_n)| < \varepsilon$. So by the definition of $\lim_{x\to\infty} F(x) = L$ we have $\lim_{x\to\infty} F(x) = 1$, as claimed. $\square$

Theorem 1.5.1 (continued 3)

Proof (continued). (d) Consider the sequence $\{x_n\}$ where $x_n = x_0 + 1/n$. Then $x_n \to x_0^+$. For $n \in \mathbb{N}$, define $C_n = \{c \in \mathcal{C} \mid X(c) \leq x_n\} = \{X \leq x_n\}$. Then the sequence of sets $\{C_n\}$ is nonincreasing and $\lim_{n\to\infty} C_n = \cap_{n=1}^{\infty} C_n = \{X \leq x_0\}$. (Notice that the text book makes an error here by simply assuming that $x_n \to x_0^+$; if any $x_n < x_0$ then they do not have $\cap_{n=1}^{\infty} C_n = \{X \leq x_0\}$, and, unless $\{x_n\}$ is a monotone decreasing sequence, they do not have that $\{C_n\}$ is a nonincreasing sequence of sets.)

Theorem 1.5.1 (continued 3)

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$$\lim_{n\to\infty} P(X \leq x_n) = \lim_{n\to\infty} P(C_n) = P\left(\lim_{n\to\infty} C_n\right)$$

$$= P\left(\cap_{n=1}^{\infty} C_n\right) = P(X \leq x_0) = F(x_0).$$

We now need to replace the limit of the sequence with the limit of the probability function.

Theorem 1.5.1 (continued 3)

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Theorem 1.5.1 (continued 4)

Theorem 1.5.1. Let X be a random variable with cumulative distribution function $F(x)$. Then

(d) $\lim_{x \downarrow x_0} F(x) = \lim_{x \to x_0^+} F(x) = F(x_0)$ (i.e., F is right continuous).

Proof (continued). Let $\varepsilon > 0$. Then there is $N \in \mathbb{N}$ such that for all $n \geq N$ we have $|P(X \leq x_n) - F(x_0)| < \varepsilon$, or $|F(x_n) - F(x_0)| < \varepsilon$. By part (a), $F(x_n) \geq F(x_0)$ so we have $0 \leq F(x_n) - F(x_0) < \varepsilon$ for all $n \geq N$. Let $x \in \mathbb{R}$ satisfy $x \geq x_0$ and $|x - x_0| = x - x_0 < \delta = 1/N$. Then $x_0 \leq x < x_0 + 1/N = x_N$ and by part (a), $F(x_0) \leq F(x) \leq F(x_0 + 1/N)$. Hence

$$0 \leq F(x) - F(x_0) \leq F(x_0 + 1/N) - F(x_0) < \varepsilon.$$

So for $\varepsilon > 0$ there is $\delta > 0$ such that if $|x - x_0| < \delta$ then $0 \leq F(x) = F(x_0) < \varepsilon$. That is, by definition, $\lim_{x \to x_0^+} F(x) - F(x_0)$, as claimed. □

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Exercise 1.5.2

Theorem 1.5.2. Let X be a random variable with cumulative distribution function F_X. Then for $a < b$ we have $P(a < X) \leq b) = F_X(b) - F_X(a)$.

Proof. Since $\{X \leq b\} = \{X \leq a\} \cup \{a < X \leq b\}$, then by finite additivity of probability, $P(X \leq b) = P(X \leq a) + P(a < X \leq b)$ or

$$P(a < X \leq b) = P(X \leq b) - P(X \leq a) = F_X(b) - F_X(a),$$

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Theorem 1.5.3

Theorem 1.5.3. For random variable X, $P(X = x) = F_X(x) - F_X(x^-)$ for all $x \in \mathbb{R}$, where $F_X(x^-) = \lim_{z \to x^-} F_X(z)$.

Proof. For any $x \in \mathbb{R}$ we have $\{x\} = \cap_{n=1}^{\infty}(x - 1/n, x]$ and the sequence of sets $(x - 1/N, x]$ is nonincreasing so that $\lim_{n\to\infty}(x - 1/N, x] = \{x\}$. So by the Continuity of the Probability Functions, Theorem 1.3.6,

$$\begin{aligned} P(X = x) &= P\left(\lim_{n\to\infty}(x - 1/n, x]\right) = \lim_{n\to\infty} P((x - 1/n, x]) \\ &= \lim_{n\to\infty}(F_X(x) - F_X(x - 1/n)) \text{ by Theorem 1.5.2} \\ &= F_X(x) - \lim_{n\to\infty} F_X(x - 1/x). \qquad (*) \end{aligned}$$

We now need to replace the limit of the sequence with the limit of the cumulative density function.

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Theorem 1.5.3. For random variable X, $P(X = x) = F_X(x) - F_X(x^-)$ for all $x \in \mathbb{R}$, where $F_X(x^-) = \lim_{z \to x^-} F_X(z)$.

Proof (continued). Since F_X is monotone then $\lim_{z \to x^-} F_X(z)$ exists for all $x \in \mathbb{R}$ (see my Analysis 1 notes on 4.2. Monotone and Inverse Functions, Theorem 4-13). Let $\varepsilon > 0$. Since $\lim_{z \to x^-} F_X(z)$ exists, say $\lim_{z \to x^-} F_X(z) = L$, then there exists $\delta > 0$ such that if $0 < x - z < \delta$ then $|F_X(x) - L| < \varepsilon$.

Theorem 1.5.3 (continued)

Theorem 1.5.3. For random variable X, $P(X = x) = F_X(x) - F_X(x^-)$ for all $x \in \mathbb{R}$, where $F_X(x^-) = \lim_{z \to x^-} F_X(z)$.

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Theorem 1.5.3 (continued)

Theorem 1.5.3. For random variable X, $P(X = x) = F_X(x) - F_X(x^-)$ for all $x \in \mathbb{R}$, where $F_X(x^-) = \lim_{z \to x^-} F_X(z)$.

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