Chapter 1. Introduction to Probability
1.5. Random Variables—Proofs of Theorems
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Theorem 1.5.1

**Theorem 1.5.1.** Let $X$ be a random variable with cumulative distribution function $F(x)$. Then

(a) For all $a$ and $b$, if $a < b$ then $F(a) \leq F(b)$ (i.e., $F$ is nondecreasing).

(b) $\lim_{x \to -\infty} F(x) = 0$.

(c) $\lim_{x \to \infty} F(x) = 1$.

(d) $\lim_{x \downarrow x_0} F(x) = \lim_{x \to x_0^+} F(x) = F(x_0)$ (i.e., $F$ is right continuous).

**Proof.** (a) With $a < b$ we have

$$\{X \leq a\} = \{x \in C \mid X(c) \leq a\} \subset \{c \in C \mid X(c) \leq b\} = \{X \leq b\},$$

so by monotonicity of $P$, Theorem 1.3.3, we have $P(X \leq a) \leq P(X \leq b)$ or $F(A) = P(X \leq a) \leq P(X \leq b) = F(b)$, as claimed.
**Theorem 1.5.1.** Let $X$ be a random variable with cumulative distribution function $F(x)$. Then

(a) For all $a$ and $b$, if $a < b$ then $F(a) \leq F(b)$ (i.e., $F$ is nondecreasing).

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**Proof.** (a) With $a < b$ we have

$$\{X \leq a\} = \{x \in C \mid X(c) \leq a\} \subset \{c \in C \mid X(c) \leq b\} = \{X \leq b\},$$

so by monotonicity of $P$, Theorem 1.3.3, we have $P(X \leq a) \leq P(X \leq b)$ or $F(A) = P(X \leq a) \leq P(X \leq b) = F(b)$, as claimed.
Theorem 1.5.1 (continued 1)

**Proof (continued).** *(b)* (This is part of Exercise 1.5.10.) For \( n \in \mathbb{N} \), define \( x_n = -n \) and define \( C_n = \{ c \in C \mid X(c) \leq -n \} = \{ X \leq -n \} \). So \( x_n \to -\infty \), the sequence of sets \( \{ C_n \} \) is nonincreasing and \( \lim_{n \to \infty} C_n = \bigcap_{n=1}^{\infty} C_n = \emptyset \). So by the Continuity of the Probability Function, Theorem 1.3.6, we have

\[
\lim_{n \to \infty} P(X \leq x_n) = \lim_{n \to \infty} P(C_n) = P \left( \lim_{n \to \infty} C_n \right) = P(\emptyset) = 0,
\]

where \( P(\emptyset) = 0 \) by Theorem 1.3.2. We now need to replace the limit of the sequence with the limit of the probability function.

Let \( \emptyset > 0 \). Then there exists \( N \in \mathbb{N} \) such that for all \( n \geq N \) we have \( |P(X \leq x_n) - 0| < \varepsilon \) or \( 0 \leq P(X \leq x_n) < \varepsilon \). Let \( x \in \mathbb{R} \) where \( x \leq -N \). By part (a), \( 0 \leq F(x) \leq F(-N) = P(X \leq x_n) < \varepsilon \). So for each \( \varepsilon > 0 \) there exists \( N \in \mathbb{N} \) such that if \( x \leq -N \) then \( |F(x) - 0| < \varepsilon \). So by the definition of \( \lim_{x \to -\infty} F(x) = L \), we have \( \lim_{x \to -\infty} F(x) = 0 \), as claimed.
Theorem 1.5.1 (continued 1)

Proof (continued). (b) (This is part of Exercise 1.5.10.) For $n \in \mathbb{N}$, define $x_n = -n$ and define $C_n = \{ c \in \mathcal{C} \mid X(c) \leq -n \} = \{ X \leq -n \}$. So $x_n \to -\infty$, the sequence of sets $\{ C_n \}$ is nonincreasing and

$$\lim_{n \to \infty} C_n = \bigcap_{n=1}^{\infty} C_n = \emptyset.$$ 

So by the Continuity of the Probability Function, Theorem 1.3.6, we have

$$\lim_{n \to \infty} P(X \leq x_n) = \lim_{n \to \infty} P(C_n) = P \left( \lim_{n \to \infty} C_n \right) = P(\emptyset) = 0,$$

where $P(\emptyset) = 0$ by Theorem 1.3.2. We now need to replace the limit of the sequence with the limit of the probability function.

Let $\emptyset > 0$. Then there exists $N \in \mathbb{N}$ such that for all $n \geq N$ we have $|P(X \leq x_n) - 0| < \varepsilon$ or $0 \leq P(X \leq x_n) < \varepsilon$. Let $x \in \mathbb{R}$ where $x \leq -N$. By part (a), $0 \leq F(x) \leq F(-N) = P(X \leq x_n) < \varepsilon$. So for each $\varepsilon > 0$ there exists $N \in \mathbb{N}$ such that if $x \leq -N$ then $|F(x) - 0| < \varepsilon$. So by the definition of $\lim_{x \to -\infty} F(x) = L$, we have $\lim_{x \to -\infty} F(x) = 0$, as claimed. \qed
Theorem 1.5.1 (continued 2)

**Proof (continued). (c)** (This is part of Exercise 1.5.10.) For \( n \in \mathbb{N} \), define \( x_n = n \) and define \( C_n = \{ c \in C \mid X(c) \leq n \} = \{ X \leq n \} \). So \( x_n \to \infty \), the sequence of sets \( \{ C_n \} \) is nondecreasing and \( \lim_{n \to \infty} C_n = \bigcup_{n=1}^{\infty} C_n = \mathbb{R} \). So by the Continuity of the Probability Function, Theorem 1.3.5, we have

\[
\lim_{n \to \infty} P(X \leq x_n) = \lim_{n \to \infty} P(C_n) = P \left( \lim_{n \to \infty} C_n \right) = P(\mathbb{R}) = 1,
\]

where \( P(\mathbb{R}) = 1 \) by Definition 1.3.1(2). We now need to replace the limit of the sequence with the limit of the probability function.

Let \( \varepsilon > 0 \). Then there exists \( N \in \mathbb{N} \) such that for all \( n \geq N \) we have

\[
|P(X \leq x_n) - 1| < \varepsilon, \text{ or } 1 - \varepsilon < P(X \leq x_n) \leq 1, \text{ or } 1 - \varepsilon < F(x_n) \leq 1.
\]

Let \( x \in \mathbb{R} \) where \( x \geq N \). By part (a), \( F(x) \geq F(N) = F(x_n) \) or \( 1 - \varepsilon < F(x_n) \leq F(x) \leq 1 \), or \( |F(x) - F(x_n)| < \varepsilon \). So for each \( \varepsilon > 0 \) there exists \( N \in \mathbb{N} \) such that if \( x \geq N \) then \( |F(x) - F(x_n)| < \varepsilon \). So by the definition of \( \lim_{x \to \infty} F(x) = L \) we have \( \lim_{x \to \infty} F(x) = 1 \), as claimed. \( \square \)
Theorem 1.5.1 (continued 2)

Proof (continued). (c) (This is part of Exercise 1.5.10.) For \( n \in \mathbb{N} \), define \( x_n = n \) and define \( C_n = \{ c \in \mathcal{C} \mid X(c) \leq n \} = \{ X \leq n \} \). So \( x_n \to \infty \), the sequence of sets \( \{ C_n \} \) is nondecreasing and \( \lim_{n \to \infty} C_n = \bigcup_{n=1}^{\infty} C_n = \mathbb{R} \). So by the Continuity of the Probability Function, Theorem 1.3.5, we have

\[
\lim_{n \to \infty} P(X \leq x_n) = \lim_{n \to \infty} P(C_n) = P \left( \lim_{n \to \infty} C_n \right) = P(\mathbb{R}) = 1,
\]

where \( P(\mathbb{R}) = 1 \) by Definition 1.3.1(2). We now need to replace the limit of the sequence with the limit of the probability function.

Let \( \varepsilon > 0 \). Then there exists \( N \in \mathbb{N} \) such that for all \( n \geq N \) we have

\[
|P(X \leq x_n) = 1| < \varepsilon, \text{ or } 1 - \varepsilon < P(X \leq x_n) \leq 1, \text{ or } 1 - \varepsilon < F(x_n) \leq 1.
\]

Let \( x \in \mathbb{R} \) where \( x \geq N \). By part (a), \( F(x) \geq F(N) = F(x_n) \) or \( 1 - \varepsilon < F(x_n) \leq F(x) \leq 1, \) or \( |F(x) - F(x_n)| < \varepsilon \). So for each \( \varepsilon > 0 \) there exists \( N \in \mathbb{N} \) such that if \( x \geq N \) then \( |F(x) - F(x_n)| < \varepsilon \). So by the definition of \( \lim_{x \to \infty} F(x) = L \) we have \( \lim_{x \to \infty} F(x) = 1 \), as claimed. \( \square \)
Proof (continued). (d) Consider the sequence \( \{x_n\} \) where \( x_n = x_0 + 1/n \). Then \( x_n \to x_0^+ \). For \( n \in \mathbb{N} \), define \( C_n = \{ c \in C \mid X(c) \leq x_n \} = \{ X \leq x_n \} \). Then the sequence of sets \( \{C_n\} \) is nonincreasing and

\[
\lim_{n \to \infty} C_n = \bigcap_{n=1}^{\infty} C_n = \{ X \leq x_0 \}.
\]

(Notice that the text book makes an error here by simply assuming that \( x_n \to x_0^+ \); if any \( x_n < x_0 \) then they do not have \( \bigcap_{n=1}^{\infty} C_n = \{ X \leq x_0 \} \), and, unless \( \{x_n\} \) is a monotone decreasing sequence, they do not have that \( \{C_n\} \) is a nonincreasing sequence of sets.)
Theorem 1.5.1 (continued 3)

**Proof (continued).** (d) Consider the sequence \( \{x_n\} \) where \( x_n = x_0 + 1/n \). Then \( x_n \to x_0^+ \). For \( n \in \mathbb{N} \), define \( C_n = \{ c \in C \mid X(c) \leq x_n \} = \{ X \leq x_n \} \). Then the sequence of sets \( \{C_n\} \) is nonincreasing and 
\[
\lim_{n \to \infty} C_n = \bigcap_{n=1}^{\infty} C_n = \{ X \leq x_0 \}. 
\]
(Notice that the text book makes an error here by simply assuming that \( x_n \to x_0^+ \); if any \( x_n < x_0 \) then they do not have \( \bigcap_{n=1}^{\infty} C_n = \{ X \leq x_0 \} \), and, unless \( \{x_n\} \) is a monotone decreasing sequence, they do not have that \( \{C_n\} \) is a nonincreasing sequence of sets.)

So the Continuity of the Probability Function, Theorem 1.3.6, we have
\[
\lim_{n \to \infty} P(X \leq x_n) = \lim_{n \to \infty} P(C_n) = P \left( \lim_{n \to \infty} C_n \right) 
= P \left( \bigcap_{n=1}^{\infty} C_n \right) = P(X \leq x_0) = F(x_0). 
\]

We now need to replace the limit of the sequence with the limit of the probability function.
Theorem 1.5.1 (continued 3)

Proof (continued). (d) Consider the sequence \( \{x_n\} \) where \( x_n = x_0 + 1/n \). Then \( x_n \to x_0^+ \). For \( n \in \mathbb{N} \), define \( C_n = \{ c \in C \mid X(c) \leq x_n \} = \{ X \leq x_n \} \).

Then the sequence of sets \( \{C_n\} \) is nonincreasing and
\[
\lim_{n \to \infty} C_n = \bigcap_{n=1}^{\infty} C_n = \{ X \leq x_0 \}. \tag{\text{Notice that the text book makes an error here by simply assuming that } x_n \to x_0^+ ; \text{ if any } x_n < x_0 \text{ then they do not have } \bigcap_{n=1}^{\infty} C_n = \{ X \leq x_0 \}, \text{ and, unless } \{x_n\} \text{ is a monotone decreasing sequence, they do not have that } \{C_n\} \text{ is a nonincreasing sequence of sets.)}
\]

So the the Continuity of the Probability Function, Theorem 1.3.6, we have
\[
\lim_{n \to \infty} P(X \leq x_n) = \lim_{n \to \infty} P(C_n) = P \left( \lim_{n \to \infty} C_n \right) \]
\[
= P \left( \bigcap_{n=1}^{\infty} C_n \right) = P(X \leq x_0) = F(x_0).\]

We now need to replace the limit of the sequence with the limit of the probability function.
Theorem 1.5.1. Let $X$ be a random variable with cumulative distribution function $F(x)$. Then

$$(d) \lim_{x \downarrow x_0} F(x) = \lim_{x \to x_0^+} F(x) = F(x_0) \text{ (i.e., } F \text{ is right continuous).}$$

Proof (continued). Let $\varepsilon > 0$. Then there is $N \in \mathbb{N}$ such that for all $n \geq N$ we have $|P(X \leq x_n) - F(x_0)| < \varepsilon$, or $|F(x_n) - F(x_0)| < \varepsilon$. By part (a), $F(x_n) \geq F(x_0)$ so we have $0 \leq F(x_n) - F(x_0) < \varepsilon$ for all $n \geq N$. Let $x \in \mathbb{R}$ satisfy $x \geq x_0$ and $|x - x_0| = x - x_0 < \delta = 1/N$. Then $x_0 \leq x < x_0 + 1/N = x_N$ and by part (a), $F(x_0) \leq F(x) \leq F(x_0 + 1/N)$. Hence

$$0 \leq F(x) - F(x_0) \leq F(x_0 + 1/N) - F(x_0) < \varepsilon.$$

So for $\varepsilon > 0$ there is $\delta > 0$ such that if $|x - x_0| < \delta$ then $0 \leq F(x) = F(x_0) < \varepsilon$. That is, by definition, $\lim_{x \to x_0^+} F(x) - F(x_0)$, as claimed. \qed
Theorem 1.5.1 (continued 4)

**Theorem 1.5.1.** Let $X$ be a random variable with cumulative distribution function $F(x)$. Then

$$(d) \lim_{x \downarrow x_0} F(x) = \lim_{x \to x_0^+} F(x) = F(x_0)$$

(i.e., $F$ is right continuous).

**Proof (continued).** Let $\varepsilon > 0$. Then there is $N \in \mathbb{N}$ such that for all $n \geq N$ we have $|P(X \leq x_n) - F(x_0)| < \varepsilon$, or $|F(x_n) - F(x_0)| < \varepsilon$. By part (a), $F(x_n) \geq F(x_0)$ so we have $0 \leq F(x_n) - F(x_0) < \varepsilon$ for all $n \geq N$. Let $x \in \mathbb{R}$ satisfy $x \geq x_0$ and $|x - x_0| = x - x_0 < \delta = 1/N$. Then $x_0 \leq x < x_0 + 1/N = x_N$ and by part (a), $F(x_0) \leq F(x) \leq F(x_0 + 1/N)$. Hence

$$0 \leq F(x) - F(x_0) \leq F(x_0 + 1/N) - F(x_0) < \varepsilon.$$  

So for $\varepsilon > 0$ there is $\delta > 0$ such that if $|x - x_0| < \delta$ then $0 \leq F(x) = F(x_0) < \varepsilon$. That is, by definition, $\lim_{x \to x_0^+} F(x) - F(x_0)$, as claimed.
Theorem 1.5.2. Let $X$ be a random variable with cumulative distribution function $F_X$. Then for $a < b$ we have $P(a < X \leq b) = F_X(b) - F_X(a)$.

Proof. Since $\{X \leq b\} = \{X \leq a\} \cup \{a < X \leq b\}$, then by finite additivity of probability, $P(X \leq b) = P(X \leq a) + P(a < X \leq b)$ or

$$P(a < X \leq b) = P(X \leq b) - P(X \leq a) = F_X(b) - F_X(a),$$

as claimed.
Theorem 1.5.2. Let $X$ be a random variable with cumulative distribution function $F_X$. Then for $a < b$ we have $P(a < X) \leq b) = F_X(b) - F_X(a)$.

Proof. Since $\{X \leq b\} = \{X \leq a\} \cup \{a < X \leq b\}$, then by finite additivity of probability, $P(X \leq b) = P(X \leq a) + P(a < X \leq b)$ or

$$P(a < X \leq b) = P(X \leq b) - P(X \leq a) = F_X(b) - F_X(a),$$

as claimed.
Theorem 1.5.3. For random variable $X$, $P(X = x) = F_X(x) - F_X(x^-)$ for all $x \in \mathbb{R}$, where $F_X(x^-) = \lim_{z \to x^-} F_X(z)$.

Proof. For any $x \in \mathbb{R}$ we have $\{x\} = \bigcap_{n=1}^{\infty} (x - 1/n, x]$ and the sequence of sets $(x - 1/N, x]$ is nonincreasing so that $\lim_{n \to \infty} (x - 1/N, x] = \{x\}$. So by the Continuity of the Probability Functions, Theorem 1.3.6,

$$P(X = x) = P\left( \lim_{n \to \infty} (x - 1/n, x] \right) = \lim_{n \to \infty} P((x - 1/n, x])$$

$$= \lim_{n \to \infty} (F_X(x) - F_X(x - 1/n)) \text{ by Theorem 1.5.2}$$

$$= F_X(x) - \lim_{n \to \infty} F_X(x - 1/n). \quad (*)$$

We now need to replace the limit of the sequence with the limit of the cumulative density function.
Theorem 1.5.3

**Theorem 1.5.3.** For random variable $X$, $P(X = x) = F_X(x) - F_X(x^-)$ for all $x \in \mathbb{R}$, where $F_X(x^-) = \lim_{z \to x^-} F_X(z)$.

**Proof.** For any $x \in \mathbb{R}$ we have $\{x\} = \bigcap_{n=1}^{\infty} (x - 1/n, x]$ and the sequence of sets $(x - 1/N, x]$ is nonincreasing so that $\lim_{n \to \infty} (x - 1/N, x] = \{x\}$. So by the Continuity of the Probability Functions, Theorem 1.3.6,

\[
P(X = x) = P \left( \lim_{n \to \infty} (x - 1/n, x] \right) = \lim_{n \to \infty} P((x - 1/n, x])
\]

\[
= \lim_{n \to \infty} (F_X(x) - F_X(x - 1/n)) \text{ by Theorem 1.5.2}
\]

\[
= F_X(x) - \lim_{n \to \infty} F_X(x - 1/n).
\]

We now need to replace the limit of the sequence with the limit of the cumulative density function.

\[
(*)
\]
Theorem 1.5.3. For random variable $X$, $P(X = x) = F_X(x) - F_X(x^-)$ for all $x \in \mathbb{R}$, where $F_X(x^-) = \lim_{z \to x^-} F_X(z)$.

Proof (continued). Since $F_X$ is monotone then $\lim_{z \to x^-} F_X(z)$ exists for all $x \in \mathbb{R}$ (see my Analysis 1 notes on 4.2. Monotone and Inverse Functions, Theorem 4-13). Let $\varepsilon > 0$. Since $\lim_{z \to x^-} F_X(z)$ exists, say $\lim_{z \to x^-} F_X(z) = L$, then there exists $\delta > 0$ such that if $0 < x - z < \delta$ then $|F_X(x) - L| < \varepsilon$. 
Theorem 1.5.3. For random variable $X$, $P(X = x) = F_X(x) - F_X(x^-)$ for all $x \in \mathbb{R}$, where $F_X(x^-) = \lim_{z \to x^-} F_X(z)$.

Proof (continued). Since $F_X$ is monotone then $\lim_{z \to x^-} F_X(z)$ exists for all $x \in \mathbb{R}$ (see my Analysis 1 notes on 4.2. Monotone and Inverse Functions, Theorem 4-13). Let $\varepsilon > 0$. Since $\lim_{z \to x^-} F_X(z)$ exists, say $\lim_{z \to x^-} F_X(z) = L$, then there exists $\delta > 0$ such that if $0 < x - z < \delta$ then $|F_X(x) - L| < \varepsilon$. Let $N \in \mathbb{N}$ satisfy $N > 1/\delta$. Then for $n \geq N$ we have $0 < x - (x - 1/n) = 1/n \leq 1/N < \delta$ and so $|F_X(x - 1/n) - L| < \varepsilon$. Therefore, by the definition of $\lim_{n \to \infty} F_X(x - 1/n) = L$, we have $\lim_{n \to \infty} F_X(x - 1/n) = L = \lim_{z \to x^-} F_X(z)$. Combining this with (*) gives $P(X = x) = F_X(x) - \lim_{z \to x^-} F_X(z) = F_X(x) - F_X(x^-)$, as claimed. \qed
Theorem 1.5.3. For random variable $X$, $P(X = x) = F_X(x) - F_X(x^-)$ for all $x \in \mathbb{R}$, where $F_X(x^-) = \lim_{z \to x^-} F_X(z)$.

Proof (continued). Since $F_X$ is monotone then $\lim_{z \to x^-} F_X(z)$ exists for all $x \in \mathbb{R}$ (see my Analysis 1 notes on 4.2. Monotone and Inverse Functions, Theorem 4-13). Let $\varepsilon > 0$. Since $\lim_{z \to x^-} F_X(z)$ exists, say $\lim_{z \to x^-} F_X(z) = L$, then there exists $\delta > 0$ such that if $0 < x - z < \delta$ then $|F_X(x) - L| < \varepsilon$. Let $N \in \mathbb{N}$ satisfy $N > 1/\delta$. Then for $n \geq N$ we have $0 < x - (x - 1/n) = 1/n \leq 1/N < \delta$ and so $|F_X(x - 1/n) - L| < \varepsilon$. Therefore, by the definition of $\lim_{n \to \infty} F_X(x - 1/n) = L$, we have $\lim_{n \to \infty} F_X(x - 1/n) = L = \lim_{z \to x^-} F_X(z)$. Combining this with (*) gives $P(X = x) = F_X(x) - \lim_{z \to x^-} F_X(z) = F_X(x) - F_X(x^-)$, as claimed. \qed