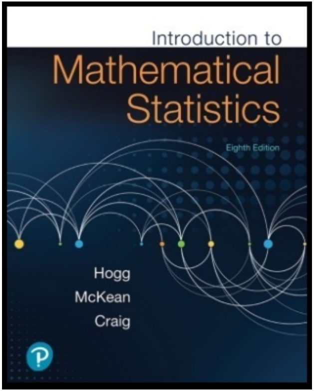
Mathematical Statistics 1

Chapter 1. Introduction to Probability

1.7. Continuous Random Variables—Proofs of Theorems



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Theorem 1.7.1

Theorem 1.7.1. The Cumulative Distribution Function Technique. Let X be a continuous random variable with probability density function f_X and support $\mathcal{S}_X$. Let $Y = g(X)$ where g is a one-to-one differentiable function on the $x = g^{-1}(y)$ and let $dx/dy = \frac{d}{dy}[g^{-1}(y)]$. Then the probability density function of Y is given by $f_Y(y) = f_X(g^{-1}(y))|dx/dy|$ for $y \in \mathcal{S}_Y$ where the support of Y is the set $\mathcal{S}_Y = \{y = g(x) \mid x \in \mathcal{S}_X\}$.

Proof. Since g is one-to-one and continuous then it is either strictly monotonically increasing or strictly monotonically decreasing (if it is monotonically increasing or decreasing but not strictly so, then it violates the property of one-to-one; if it is a mixture of strictly monotone increasing and strictly monotone decreasing then it violates the property of one-to-one by the Intermediate Value Theorem).

Theorem 1.7.1

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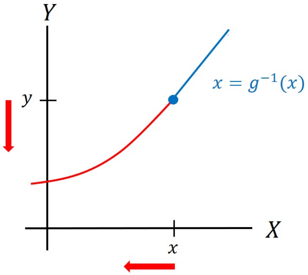
Theorem 1.7.1 (continued 1)

Proof (continued). First, suppose g is strictly monotone increasing:



Notice that since g is hypothesized to be differentiable, then g^{-1} is also differentiable. The cumulative distribution function of Y is

$$F_Y(y) = P(Y \leq y) = P(g(X) \leq y) = P(X \leq g(y)) = F_X(g^{-1}(y)).$$

Theorem 1.7.1 (continued 2)

Proof (continued). Hence, the probability distribution function of Y is

$$\begin{aligned} f_Y(y) &= \frac{d}{dy}[F_Y(y)] = \frac{d}{dy}[F_X(g^{-1}(y))] \\ &= \left(\frac{d}{dx}[F_X(x)]\frac{dx}{dy}\right)\bigg|_{x=g^{-1}(y)} \text{ by the Chain Rule} \\ &\quad \text{and since } x = g^{-1}(y) \text{ is a differentiable function by hypothesis} \\ &= \left(f_X(x)\frac{dx}{dy}\right)\bigg|_{x=g^{-1}(y)} = f_X(g^{-1}(y))\frac{dx}{dy}. \end{aligned}$$

Since $y = g(x)$ is strictly increasing in this case then g^{-1} is strictly monotone increasing (see my online Analysis 1 [MATH 4217/5217] notes on 4-2. Monotone and Inverse Functions; see Theorem 4-16. Continuity of the Inverse Function) and $dx/dy = d[g^{-1}(y)]dy > 0$ so that $|dx/dy| = dx/dy$. Therefore $f_Y(y) = f_X(g^{-1}(y))|dx/dy|$, as claimed.

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Proof (continued). Hence, the probability distribution function of Y is

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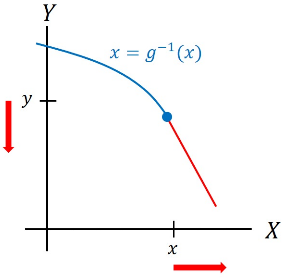
Theorem 1.7.1 (continued 3)

Proof (continued). First, suppose g is strictly monotone decreasing:



The cumulative distribution function of Y is

$$F_Y(y) = P(Y \leq y) = P(g(X) \leq y) = P(X \geq x)$$

$$= 1 - P(X \leq x) = 1 - F_X(x) = 1 - F_X(g^{-1}(y)).$$

Theorem 1.7.1 (continued 4)

Proof (continued). Hence, the probability distribution function of Y is

$$\begin{aligned} f_Y(y) &= \frac{d}{dy}[F_Y(y)] = \frac{d}{dy}[1 - F_X(g^{-1}(y))] \\ &= \left(-\frac{d}{dx}[F_X(x)]\frac{dx}{dy} \right)\bigg|_{x=g^{-1}(y)} \text{ by the Chain Rule} \\ &= \left(f_X(x)\frac{-dx}{dy} \right)\bigg|_{x=g^{-1}(y)} = f_X(g^{-1}(y))\left(-\frac{dx}{dy} \right). \end{aligned}$$

Since $y = g(x)$ is strictly decreasing in this case then g^{-1} is strictly monotone decreasing and $dx/dy = d[g^{-1}(y)]dy < 0$ so that $|dx/dy| = -dx/dy$. Therefore $f_Y(y) = f_X(g^{-1}(y))|dx/dy|$, as claimed. □

Theorem 1.7.1 (continued 4)

Proof (continued). Hence, the probability distribution function of Y is

$$\begin{aligned} f_Y(y) &= \frac{d}{dy}[F_Y(y)] = \frac{d}{dy}[1 - F_X(g^{-1}(y))] \\ &= \left(-\frac{d}{dx}[F_X(x)]\frac{dx}{dy}\right)\bigg|_{x=g^{-1}(y)} \text{ by the Chain Rule} \\ &= \left(f_X(x)\frac{-dx}{dy}\right)\bigg|_{x=g^{-1}(y)} = f_X(g^{-1}(y))\left(-\frac{dx}{dy}\right). \end{aligned}$$

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Exercise 1.7.24

Exercise 1.7.24. Let X have the uniform probability density function $f_X(x) = 1/\pi$ for $-\pi/2 < x < \pi/2$. Find the probability density function of $Y = \tan(X)$. This is the probability density function of a *Cauchy distribution*.

Solution. Notice that here we have $g(x) = \tan(x)$ which is one-to-one and differentiable on $\mathcal{S}_X = (-\pi/2, \pi/2)$. We follow the algorithm described above.

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2. With $y = g(x) = \tan(x)$ and $x \in (-\pi/2, \pi/2)$, $x = g^{-1}(y) = \tan^{-1}(y)$. Notice that $-\infty < y < \infty$.

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4. The probability density function of Y is

$$f_Y(y) = f_X(g^{-1}(y)) \left| \frac{dx}{dx} \right| = f_X(\tan^{-1}(y)) \left| \frac{dx}{dy} \right| = \frac{1}{\pi} \frac{1}{1+y^2}$$

where $-\infty < y < \infty$. □

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