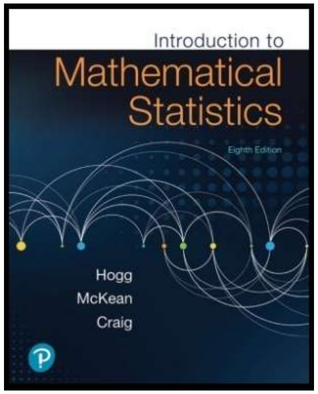
Mathematical Statistics 1

Chapter 1. Introduction to Probability

1.9. Some Special Expectations—Proofs of Theorems



Theorem 1.9.1

Theorem 1.9.1. Let X be a random variable with finite mean μ and finite variance σ^2. Then for all constants a and b we have $\mathrm{Var}(aX+b) = a^2\mathrm{Var}(X)$.

Proof. Since E is linear by Theorem 1.8.2, then $E[aX+b] = aE[X]+b = a\mu+b$. So

$$\begin{aligned}\mathrm{Var}(aX+b) &= E[((aX+b)-(a\mu+b))^2] = E[a^2(X-\mu)^2] \\ &= a^2E[(X-\mu)^2] = a^2\mathrm{Var}(X),\end{aligned}$$

as claimed. □

Exercise 1.9.3(a)

Exercise 1.9.3(a). Let X have distribution $f(x) = 6x(1-x) = 6x-6x^2$, $0 < x < 1$, zero elsewhere. Compute $P(\mu - 2\sigma < X < \mu + s\sigma)$.

Solution. First,

$$\begin{aligned}\mu = E[X] &= \int_{-\infty}^{\infty} xf(x)\,dx = \int_0^1 x(6x-6x^2)\,dx \\ &= \int_0^1 (6x^2-6x^3)\,dx = \left.\left(2x^3 - \frac{3}{2}x^4\right)\right|_0^1 = \frac{1}{2}.\end{aligned}$$

Next, for σ^2 we first find $E[X]$:

$$\begin{aligned}E[X^2] &= \int_{-\infty}^{\infty} x^2f(x)\,dx = \int_0^1 x^2(6x-6x^2)\,dx = \int_0^1 (6x^3-6x^4)\,dx \\ &= \left.\left(\frac{3}{2}x^4 - \frac{6}{5}x^5\right)\right|_0^1 = \frac{3}{2} - \frac{6}{5} = \frac{3}{10}.\end{aligned}$$

So $\sigma^2 = E[X^2] - \mu^2 = 3/10 - (1/2)^2 = 1/20$ and then $\sigma = 1/\sqrt{20}$.

Exercise 1.9.3(a) (continued)

Solution (continued). Hence (since $2\sigma = 2/\sqrt{20} = 1/\sqrt{5}$)

$$\begin{aligned}P(\mu - 2\sigma < X < \mu + 2\sigma) &= P\left(\frac{1}{2} - \frac{1}{\sqrt{5}} < X < \frac{1}{2} + \frac{1}{\sqrt{5}}\right) \\ &= P\left(\frac{\sqrt{5}-2}{2\sqrt{5}} < X < \frac{\sqrt{5}+2}{2\sqrt{5}}\right) \\ &= \int_{(\sqrt{5}-2)/(2\sqrt{5})}^{(\sqrt{5}+2)/(2\sqrt{5})} (6x-6x^2)\,dx = (3x^2-2x^3)\Big|_{(\sqrt{5}-2)/(2\sqrt{5})}^{(\sqrt{5}+2)/(2\sqrt{5})} \\ &= \left(3\left(\frac{\sqrt{5}+2}{2\sqrt{5}}\right)^2 - 2\left(\frac{\sqrt{5}+2}{2\sqrt{5}}\right)^3\right) - \left(3\left(\frac{\sqrt{5}-2}{2\sqrt{5}}\right)^2 - 2\left(\frac{\sqrt{5}-2}{2\sqrt{5}}\right)^3\right) \\ &\approx 0.98387.\end{aligned}$$

□

Exercise 1.9.7.

Exercise 1.9.7. Show that the moment generating function of the random variable X having the probability density function $f(x) = 1/3$, $-1 < x < 2$, zero elsewhere, is

$$M(t) = \begin{cases} \frac{e^{2t}-e^{-t}}{3t} & \text{for } t \neq 0 \\ 1 & \text{for } t = 0. \end{cases}$$

Solution. By definition,

$$M(t) = E[e^{tX}] = \int_{-\infty}^{\infty} e^{tX} f(x)\, dx = \int_{-1}^{2} \frac{1}{3} e^{tx}\, dx$$

$$= \frac{1}{3t} e^{tx} \Big|_{-1}^{2} = \frac{e^{2t} - e^{-t}}{3t} \text{ for } t \neq 0.$$

As commented above, $M(0) = 1$ when a moment generating function exists and so the result follows. □