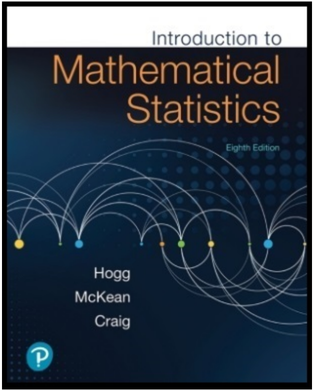
Mathematical Statistics 1

Chapter 1. Introduction to Probability

1.9. Some Special Expectations—Proofs of Theorems



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Theorem 1.9.1

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Proof. Since E is linear by Theorem 1.8.2, then $E[aX+b] = aE[X]+b = a\mu+b$. So

$$\text{Var}(aX+b) = E[((aX+b)-(a\mu+b))^2] = E[a^2(X-\mu)^2]$$
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Exercise 1.9.3(a)

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Solution. First,

$$\mu = E[X] = \int_{-\infty}^{\infty} xf(x)\,dx = \int_0^1 x(6x - 6x^2)\,dx$$
$$= \int_0^1 (6x^2 - 6x^3)\,dx = \left(2x^3 - \frac{3}{2}x^4\right)\Big|_0^1 = \frac{1}{2}.$$

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Next, for σ^2 we first find $E[X]$:

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$$= \left(\frac{3}{2}x^4 - \frac{6}{5}x^5\right)\Big|_0^1 = \frac{3}{2} - \frac{6}{5} = \frac{3}{10}.$$

So $\sigma^2 = E[X^2] - \mu^2 = 3/10 - (1/2)^2 = 1/20$ and then $\sigma = 1/\sqrt{20}$.

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Exercise 1.9.3(a) (continued)

Solution (continued). Hence (since $2\sigma = 2/\sqrt{20} = 1/\sqrt{5}$)

$$P(\mu - 2\sigma < X < \mu + 2\sigma) = P\left(\frac{1}{2} - \frac{1}{\sqrt{5}} < X < \frac{1}{2} + \frac{1}{\sqrt{5}}\right)$$

$$= P\left(\frac{\sqrt{5}-2}{2\sqrt{5}} < X < \frac{\sqrt{5}+2}{2\sqrt{5}}\right)$$

$$= \int_{(\sqrt{5}-2)/(2\sqrt{5})}^{(\sqrt{5}+2)/(2\sqrt{5})} (6x - 6x^2)\,dx = (3x^2 - 2x^3)\Big|_{(\sqrt{5}-2)/(2\sqrt{5})}^{(\sqrt{5}+2)/(2\sqrt{5})}$$

$$= \left(3\left(\frac{\sqrt{5}+2}{2\sqrt{5}}\right)^2 - 2\left(\frac{\sqrt{5}+2}{2\sqrt{5}}\right)^3\right) - \left(3\left(\frac{\sqrt{5}-2}{2\sqrt{5}}\right)^2 - 2\left(\frac{\sqrt{5}-2}{2\sqrt{5}}\right)^3\right)$$

$$\approx 0.98387. \qquad \square$$

Exercise 1.9.7.

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$$M(t) = \begin{cases} \frac{e^{2t} - e^{-t}}{3t} & \text{for } t \neq 0 \\ 1 & \text{for } t = 0. \end{cases}$$

Solution. By definition,

$$M(t) = E[e^{tX}] = \int_{-\infty}^{\infty} e^{tX} f(x)\, dx = \int_{-1}^{2} \frac{1}{3} e^{tx}\, dx$$

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