Chapter 2. Multivariate Distributions

2.1. Distributions of Two Random Variables—Proofs of Theorems

Theorem 2.1.1. Let $(X_1, X_2)$ be a random vector. Let $Y_1 = g_1(X_1, X_2)$ and $Y_2 = g_2(X_1, X_2)$ be random variables whose expectations exist. Then for all $k_1, k_2 \in \mathbb{R}$ we have

$$E[k_1 Y_1 + k_2 Y_2] = k_1 E[Y_1] + k_2 E[Y_2].$$

Proof. We give a proof for the continuous case and leave the discrete case as an exercise. By the Triangle Inequality on $\mathbb{R}$ and the linearity of integration,

$$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |k_1 g_1(x_1, x_2) + k_2 g_2(x_1, x_2)| f_{X_1, X_2}(x_1, x_2) \, dx_1 \, dx_2$$

$$\leq \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (|k_1| |g_1(x_1, x_2)| + |k_2| |g_2(x_1, x_2)|) f_{X_1, X_2}(x_1, x_2) \, dx_1 \, dx_2$$

where the boundedness follows by the hypothesis that the expectations of $Y_1 = g_1(X_1, X_2)$ and $Y_2 = g_2(X_1, X_2)$ exist. Therefore the expectation of $k_1 g_1(x_1, x_2) + k_2 g_2(x_1, x_2)$ exists. Again by linearity of integration,

$$E[k_1 Y_1 + k_2 Y_2] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (k_1 g_1(x_1, x_2) + k_2 g_2(x_1, x_2)) f_{X_1, X_2}(x_1, x_2) \, dx_1 \, dx_2$$

$$= k_1 \int_{-\infty}^{\infty} g_1(x_1, x_2) f_{X_1}(x_1) \, dx_1 \, dx_2$$

$$+ k_2 \int_{-\infty}^{\infty} g_2(x_1, x_2) f_{X_1}(x_1) \, dx_1 \, dx_2 = k_1 E[Y_1] + k_2 E[Y_2],$$

as claimed.