Chapter 2. Multivariate Distributions

2.1. Distributions of Two Random Variables—Proofs of Theorems
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\[
E[k_1 Y_1 + k_2 Y_2] = k_1 E[Y_1] + k_2 E[Y_2].
\]

Proof. We give a proof for the continuous case and leave the discrete case as an exercise. By the Triangle Inequality on \(\mathbb{R}\) and the linearity of integration

\[
\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |k_1 g_1(x_1, x_2) + k_2 g_2(x_1, x_2)| f_{X_1, X_2}(x_1, x_2) \, dx_1 \, dx_2
\]

\[
\leq \int_{-\infty}^{\infty} (|k_1||g_1(x_1, x_2)| + |k_2||g_2(x_1, x_2)|) f_{X_1, X_2}(x_1, x_2) \, dx_1 \, dx_2
\]
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\int_{-\infty}^{\infty} |k_1 g_1(x_1, x_2) + k_2 g_2(x_1, x_2)| f_{X_1, X_2}(x_1, x_2) \, dx_1 \, dx_2 \\
\leq \int_{-\infty}^{\infty} (|k_1||g_1(x_1, x_2)| + |k_2||g_2(x_1, x_2)|) f_{X_1, X_2}(x_1, x_2) \, dx_1 \, dx_2
\]
Theorem 2.1.1 (continued)

Proof.  

\[ |k_1| \int_{-\infty}^{\infty} |g_1(x_1, x_2)| f_{X_1, X_2}(x_1, x_2) \, dx_1 \, dx_2 \]

\[ + |k_2| \int_{-\infty}^{\infty} |g_2(x_1, x_2)| f_{X_1, X_2}(x_1, x_2) \, dx_1 \, dx_2 \quad < \infty \]

where the boundedness follows by the hypothesis that the expectations of \( Y_1 = g_1(X_1, X_2) \) and \( Y_2 = g_2(X_1, X_2) \) exist. Therefore the expectation of \( k_1 g_1(x_1, x_2) + k_2 g_2(x_1, x_2) \) exists. Again by linearity of integration,

\[ E[k_1 Y_1 + k_2 Y_2] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (k_1 g_1(x_1, x_2) + k_2 g_2(x_1, x_2)) f_{X_1, X_2}(x_1, x_2) \, dx_1 \, dx_2 \]

\[ = k_1 \int_{-\infty}^{\infty} g_1(x_1, x_2) f_{X_1, X_2}(x_1, x_2) \, dx_1 \, dx_2 \]

\[ + k_2 \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g_2(x_1, x_2) f_{X_1, X_2}(x_1, x_2) \, dx_1 \, dx_2 = k_1 E[Y_1] + k_2 E[Y_2], \]

as claimed.
Theorem 2.1.1 (continued)

Proof.

\[
|k_1| \int_{-\infty}^{\infty} |g_1(x_1, x_2)| f_{X_1, X_2}(x_1, x_2) \, dx_1 \, dx_2 \\
+ |k_2| \int_{-\infty}^{\infty} |g_2(x_1, x_2)| f_{X_1, X_2}(x_1, x_2) \, dx_1 \, dx_2 < \infty
\]

where the boundedness follows by the hypothesis that the expectations of \( Y_1 = g_1(X_1, X_2) \) and \( Y_2 = g_2(X_1, X_2) \) exist. Therefore the expectation of \( k_1 g_1(x_1, x_2) + k_2 g_2(x_1, x_2) \) exists. Again by linearity of integration,

\[
E[k_1 Y_1 + k_2 Y_2] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (k_1 g_1(x_1, x_2) + k_2 g_2(x_1, x_2)) f_{X_1, X_2}(x_1, x_2) \, dx_1 \, dx_2 \\
= k_1 \int_{-\infty}^{\infty} g_1(x_1, x_2) f_{X_1, X_2}(x_1, x_2) \, dx_1 \, dx_2 \\
+ k_2 \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g_2(x_1, x_2) f_{X_1, X_2}(x_1, x_2) \, dx_1 \, dx_2 = k_1 E[Y_1] + k_2 E[Y_2],
\]

as claimed.