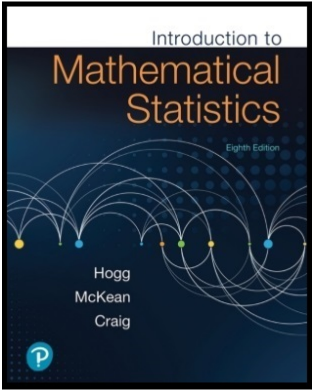
Mathematical Statistics 1

Chapter 2. Multivariate Distributions

2.1. Distributions of Two Random Variables—Proofs of Theorems



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Theorem 2.1.1

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$$E[k_1 Y_1 + k_2 Y_2] = k_1 E[Y_1] + k_2 E[Y_2].$$

Proof. We give a proof for the continuous case and leave the discrete case as an exercise. By the Triangle Inequality on $\mathbb{R}$ and the linearity of integration

$$\int_{-\infty}^{\infty} |k_1 g_1(x_1, x_2) + k_2 g_2(x_1, x_2)| f_{X_1,X_2}(x_1, x_2)\, dx_1\, dx_2$$

$$\leq \int_{-\infty}^{\infty} (|k_1||g_1(x_1, x_2)| + |k_2||g_2(x_1, x_2)|) f_{X_1,X_2}(x_1, x_2)\, dx_1\, dx_2$$

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Theorem 2.1.1 (continued)

Proof.

$$= |k_1| \int_{-\infty}^{\infty} |g_1(x_1, x_2)| f_{X_1,X_2}(x_1, x_2)\, dx_1\, dx_2$$

$$+|k_2| \int_{-\infty}^{\infty} |g_2(x_1, x_2)| f_{X_1,X_2}(x_1, x_2)\, dx_1\, dx_2 < \infty$$

where the boundedness follows by the hypothesis that the expectations of $Y_1 = g_1(X_1, X_2)$ an d $Y_2 = g_2(X_1, X_2)$ exist. Therefore the expectation of $k_1 g_1(x_1, x_2) + k_2 g_2(x_1, x_2)$ exists. Again by linearity of integration,

$$E[k_1 Y_1 + k_2 Y_2] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (k_1 g_1(x_1, x_2) + k_2 g_2(x_1, x_2)) f_{X_1,X_2}(x_1, x_2)\, dx_1\, dx_2$$

$$= k_1 \int_{-\infty}^{\infty} g_1(x_1, x_2) f_{X_1,X_2}(x_1, x_2)\, dx_1\, dx_2$$

$$+k_2 \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g_2(x_1, x_2) f_{X_1,X_2}(x_1, x_2)\, dx_1\, dx_2 = k_1 E[Y_1] + k_2 E[Y_2],$$

as claimed. □

Theorem 2.1.1 (continued)

Proof.

$$= |k_1| \int_{-\infty}^{\infty} |g_1(x_1, x_2)| f_{X_1,X_2}(x_1, x_2)\, dx_1\, dx_2$$

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$$E[k_1 Y_1 + k_2 Y_2] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (k_1 g_1(x_1, x_2) + k_2 g_2(x_1, x_2)) f_{X_1,X_2}(x_1, x_2)\, dx_1\, dx_2$$

$$= k_1 \int_{-\infty}^{\infty} g_1(x_1, x_2) f_{X_1,X_2}(x_1, x_2)\, dx_1\, dx_2$$

$$+k_2 \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g_2(x_1, x_2) f_{X_1,X_2}(x_1, x_2)\, dx_1\, dx_2 = k_1 E[Y_1] + k_2 E[Y_2],$$

as claimed. □