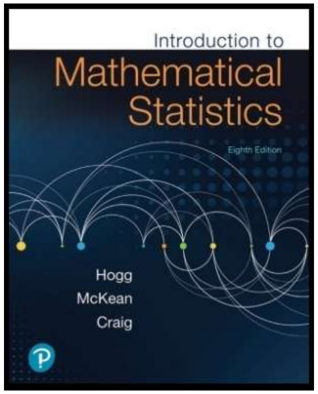
# Mathematical Statistics 1

**Chapter 2. Multivariate Distributions**

2.3. Conditional Distributions and Expectations—Proofs of Theorems








## Theorem 2.3.1

**Theorem 2.3.1.** Let $(X_1, X_2)$ be a random vector such that the variance of $X_2$ is finite. Then

(a) $E[E[X_2 \mid X_1]] = E[X_2]$, and

(b) $\mathrm{Var}([E[X_2 \mid X_1]) \le \mathrm{Var}(X_2)$.

**Proof.** We give proofs for the continuous case and leave the discrete case as an exercise.

**(a)** We have

$$\begin{aligned} E[X_2] &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_2 f(x_1,x_2)\,dx_2\,dx_1 \\ &= \int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} x_2\frac{f(x_1,x_2)}{f_1(x_1)}\,dx_2\right) f_1(x_2)\,dx_1 \\ &= \int_{-\infty}^{\infty} E[X_2 \mid x_1] f_1(x_1)\,dx_1 = E[E[X_2 \mid X_1]] \end{aligned}$$

(notice that $E[X_2 \mid x_1]$ is a function of $x_1$).





## Theorem 2.3.1 (continued 1)

**Proof (continued). (b)** Let $\mu_2 = E[X_2]$, then

$$\mathrm{Var}(X_2) = E[(X_2-\mu_2)^2] = E[(X_2 - E[X_2 \mid X_1] + E[X_2 \mid X_1] - \mu_2)^2]$$

$$\begin{aligned} &= E[(X_2 - E[X_2 \mid X_1] + E[X_2 \mid X_1] - \mu_2)(X_2 - E[X_2 \mid X_1] + E[X_2 \mid X_1] - \mu_2)] \\ &= E[(X_2 - E[X_2 \mid X_1])^2 + (E[X_2 \mid X_1] - \mu_2)(X_2 - E[X_2 \mid X_1]) \\ &\quad + (X_2 - E[X_2 \mid X_1])(E[X_2 \mid X_1] - \mu_2) + (E[X_2 \mid X_1] - \mu_2)^2] \\ &= E[(X_2 - E[X_2 \mid X_1])^2] + 2E[(E[X_2 \mid X_1] - \mu_2)(X_2 - E[X_2 \mid X_1])] \\ &\quad + E[(E[X_2 \mid X_1] - \mu_2)^2] \end{aligned}$$

where the last equality holds since $E$ is linear by Theorem 2.1.1.





## Theorem 2.3.1 (continued 2)

**Proof (continued).**

$$2E[(E[X_2 \mid X)1] - \mu_2)(X_2 - E[X_2 \mid X_1])]$$

$$\begin{aligned} &= 2\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(X_2 - E[X_2 \mid x_1])(E[X_2 \mid x_1] - \mu_2) f(x_1,x_2)\,dx_2\,dx_1 \\ &= 2\int_{-\infty}^{\infty}(E[X_2] - \mu_2)\left(\int_{-\infty}^{\infty}(x_2 - E[X_2 \mid x_1])\frac{f(x_1,x_2)}{f_1(x_1)}\,dx_2\right) f_1(x_1)\,dx_1. \end{aligned}$$

We have

$$\begin{aligned} &\int_{-\infty}^{\infty}(x_2 - E[X_2 \mid x_1])\frac{f(x_1,x_2)}{f_1(x_1)}\,dx_2 \\ &= \int_{-\infty}^{\infty} x_2\frac{f(x_1,x_2)}{f_1(x_1)}\,dx_2 - E[X_2 \mid x_1]\int_{-\infty}^{\infty}\frac{f(x_1,x_2)}{f_1(x_1)}\,dx_2 \\ &= E[X_2 \mid x_1] - EpX_2 \mid x_2](1) = 0, \end{aligned}$$

...

# Theorem 2.3.1 (continued 3)

**Proof (continued).** ...so $2E[(X_2 - E[X_2 \mid X_1])(E[X_2 \mid X_1] - \mu_2)] = 0$ and

$$\begin{aligned}\mathrm{Var}(X_2) &= E[(X_2 - E[X_2 \mid X_1])^2] + E[(E[X_2 \mid X_1] - \mu_2)^2] \\ &\geq E[(E[X_2 \mid X_1] - \mu_2)^2] \qquad (*)\end{aligned}$$

since $E[(X_2 - E[X_2 \mid X_1])^2] \geq 0$. Now for random variable $X$, $\mathrm{Var}(X) = E[(X-\mu)^2] = E[(X - E[X])^2]$ (see Definition 1.9.2), so random variable $E[X_2 \mid X_1]$ has mean $E[E[X_2 \mid X_1]]$ and by part (a), $E[E[X_2 \mid X_1]] = E[X_2] = \mu_2$ so that

$$\mathrm{Var}(E[X_2 \mid X_1]) = E[(E[X_2 \mid X_1] - \mu_2)^2]$$

and hence by $(*)$

$$\mathrm{Var}(X_2) \geq E[(E[X_2 \mid X_1] - \mu_2)^2] = \mathrm{Var}(E[X_2 \mid X_1]),$$

as claimed. □