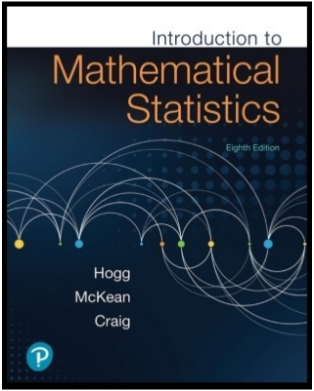
Mathematical Statistics 1

Chapter 2. Multivariate Distributions

2.4. Independent Random Variables—Proofs of Theorems



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Theorem 2.4.1

Theorem 2.4.1. Let the random variables X_1 and X_2 have supports $\mathcal{S}_1$ and $\mathcal{S}_2$, respectively, and have the joint probability density function $f(x_1, x_2)$. Then X_1 and X_2 are independent if and only if $f(x_1, x_2)$ can be written as a product of a nonnegative function of x_1 and a nonnegative function of x_2. That is, $f(x_1, x_2) \equiv g(x_1)h(x_2)$ for some $g(x_1) > 0$ for $x_1 \in \mathcal{S}_1$ and 0 elsewhere, and some $h(x_2) > 0$ for $x_2 \in \mathcal{S}_2$ and 0 elsewhere.

Proof. If X_1 and X_2 are independent then $f(x_1, x_2) = f_1(x_1)f_2(x_2)$ where f_1 and f_2 are the marginal probability density functions of X_1 and X_2 so that $f_1(x_1) > 0$ for $x_1 \in \mathcal{S}_1$ and $f_2(x_2) > 0$ for $x_2 \in \mathcal{S}_2$, as claimed.

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Conversely, if $f(x_1, x_2) \equiv g(x_1)h(x_2)$, where g and h are nonnegative and positive on $\mathcal{S}_1$ and $\mathcal{S}_2$ respectively, then for the continuous random variables we have the marginal probability density functions

$$f_1(x_1) = \int_{-\infty}^{\infty} g(x_1)f(x_2)\, dx_2 = g(x_1)\int_{-\infty}^{\infty} h(x_2)\, dx_2 = c_1 g(x_1)$$

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Proof. If X_1 and X_2 are independent then $f(x_1, x_2) = f_1(x_1)f_2(x_2)$ where f_1 and f_2 are the marginal probability density functions of X_1 and X_2 so that $f_1(x_1) > 0$ for $x_1 \in \mathcal{S}_1$ and $f_2(x_2) > 0$ for $x_2 \in \mathcal{S}_2$, as claimed.

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$$f_1(x_1) = \int_{-\infty}^{\infty} g(x_1)f(x_2)\,dx_2 = g(x_1)\int_{-\infty}^{\infty} h(x_2)\,dx_2 = c_1 g(x_1)$$

Theorem 2.4.1 (continued)

Proof (continued). . . . and

$$f_2(x_2) = \int_{-\infty}^{\infty} g(x_1)f(x_2)\,dx_1 = h(x_2)\int_{-\infty}^{\infty} g(x_1)\,dx_1 = c_2 h(x_2)$$

for some c_1 and c_2 (notice that we need g and h to be integrable here). Now

$$1 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x_1)h(x_2)\,dx_1\,dx_2 = \int_{-\infty}^{\infty} g(x_1)\,dx_1 \int_{-\infty}^{\infty} h(x_2)\,dx_2 = c_1 c_2.$$

So we have

$$f(x_1, x_2) \equiv g(x_1)h(x_2) = c_1 g(x_1) c_2 h(x_2) = f_1(x_1) f_2(x_2)$$

so that continuous random variables X_1 and X_2 are independent, as claimed. We leave the discrete case as an exercise. □

Theorem 2.4.1 (continued)

Proof (continued). ... and

$$f_2(x_2) = \int_{-\infty}^{\infty} g(x_1) f(x_2)\, dx_1 = h(x_2) \int_{-\infty}^{\infty} g(x_1)\, dx_1 = c_2 h(x_2)$$

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Theorem 2.4.2

Theorem 2.4.2. Let (X_1, X_2) be a random vector with joint cumulative distribution function $F(x_1, x_2)$ and let X_1 and X_2 have the marginal cumulative distribution functions $F_1(x_1)$ and $F_2(x_2)$, respectively. Then X_1 and X_2 are independent if and only if $F(x_1, x_2) = F_1(x_1)F_2(x_2)$ for all $(x_1, x_2) \in \mathbb{R}^2$.

Proof. We give a proof for continuous random variables and leave the discrete case as an exercise. It is shown in Note 2.1.B that the joint cumulative distribution function F_{X_1,X_2} and the probability density function f_{X_1,X_2} for random vector (X_1, X_2) satisfies

$$\frac{\partial^2[F_{X_1,X_2}(x_1, x_2)]}{\partial x_1\, \partial x_2} = f_{X_1,X_2}(x_1, x_2).$$

So if $F(x_1, x_2) = F_1(x_1)F_2(x_2)$ then...

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Proof. We give a proof for continuous random variables and leave the discrete case as an exercise. It is shown in Note 2.1.B that the joint cumulative distribution function F_{X_1,X_2} and the probability density function f_{X_1,X_2} for random vector (X_1, X_2) satisfies

$$\frac{\partial^2[F_{X_1,X_2}(x_1, x_2)]}{\partial x_1\, \partial x_2} = f_{X_1,X_2}(x_1, x_2).$$

So if $F(x_1, x_2) = F_1(x_1)F_2(x_2)$ then...

Theorem 2.4.2 (continued)

Proof (continued).

$$f(x_1, x_2) = \frac{\partial^2}{\partial x_1\, \partial x_2}[F(x_1, x_2)] = \frac{\partial^2}{\partial x_1\, \partial x_2}[F_1(x_1)F_2(x_2)]$$

$$= \frac{d}{dx_1}[F_1(x_1)]\frac{d}{dx_2}[F_2(x_2)] = f_1(x_1)f_2(x_2)$$

by Note 1.7.A, and so X_1 and X_2 are independent by Definition 2.4.1, as claimed.

Suppose X_1 and X_2 are independent so that $f(x_1, x_2) = f_1(x_1)f_2(x_2)$ by Definition 2.4.1, then by the definition of the joint cumulative distribution function

$$F(x_1, x_2) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(w_1, w_2)\, dw_2\, dw_1 = \int_{-\infty}^{x_1}\int_{-\infty}^{x_2} f_1(w_1)f_2(w_2)\, dw_2\, dw_1$$

$$= \int_{-\infty}^{x_1} f_1(w_1)\, dw_1 \int_{-\infty}^{x_2} f_2(w_2)\, dw_2 = F_1(x_1)F_2(x_2),$$

as claimed. □

Theorem 2.4.2 (continued)

Proof (continued).

$$f(x_1, x_2) = \frac{\partial^2}{\partial x_1\, \partial x_2}[F(x_1, x_2)] = \frac{\partial^2}{\partial x_1\, \partial x_2}[F_1(x_1)F_2(x_2)]$$
$$= \frac{d}{dx_1}[F_1(x_1)]\frac{d}{dx_2}[F_2(x_2)] = f_1(x_1)f_2(x_2)$$

by Note 1.7.A, and so X_1 and X_2 are independent by Definition 2.4.1, as claimed.

Suppose X_1 and X_2 are independent so that $f(x_1, x_2) = f_1(x_1)f_2(x_2)$ by Definition 2.4.1, then by the definition of the joint cumulative distribution function

$$F(x_1, x_2) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(w_1, w_2)\, dw_2\, dw_1 = \int_{-\infty}^{x_1}\int_{-\infty}^{x_2} f_1(w_1)f_2(w_2)\, dw_2\, dw_1$$
$$= \int_{-\infty}^{x_1} f_1(w_1)\, dw_1 \int_{-\infty}^{x_2} f_2(w_2)\, dw_2 = F_1(x_1)F_2(x_2),$$

as claimed. □

Theorem 2.4.3

Theorem 2.4.3. The random variables X_1 and X_2 are independent random variables if and only if

$$P(a < X_1 \leq b, c < X_2 \leq d) = P(a < X_1 \leq b)P(c < X_2 \leq d)$$

for every $a < b$ and $c < d$, where a, b, c, d are constants.

Proof. If X_1 and X_2 are independent then

$$\begin{aligned} & P(a < X_1 \leq b,\ c < X_2 \leq d) \\ = \ & F(b,d) - F(a,d) - F(b,c) + F(a,c) \text{ by Exercise 2.1.3} \\ & \text{(see Notes 2.1.A), where } F \text{ is the joint cdf of } X_1 \text{ and } X_2 \\ = \ & F_1(b)F_2(d) - F_1(a)F_2(d) - F_1(b)F_2(c) - F_1(a)F_2(c) \\ & \text{ by Theorem 2.4.2} \\ = \ & (F_1(b) - F_1(a))(F_2(d) - F_2(c)) = P(a < X_1 \leq b)P(c < X_2 \leq d), \end{aligned}$$

as claimed.

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for every $a < b$ and $c < d$, where a, b, c, d are constants.

Proof. If X_1 and X_2 are independent then

$$P(a < X_1 \leq b,\ c < X_2 \leq d)$$

$$\begin{aligned}
&= F(b,d) - F(a,d) - F(b,c) + F(a,c) \text{ by Exercise 2.1.3} \\
&\quad \text{(see Notes 2.1.A), where } F \text{ is the joint cdf of } X_1 \text{ and } X_2 \\
&= F_1(b)F_2(d) - F_1(a)F_2(d) - F_1(b)F_2(c) - F_1(a)F_2(c) \\
&\quad \text{ by Theorem 2.4.2} \\
&= (F_1(b) - F_1(a))(F_2(d) - F_2(c)) = P(a < X_1 \leq b)P(c < X_2 \leq d),
\end{aligned}$$

as claimed.

Theorem 2.4.3 (continued)

Theorem 2.4.3. The random variables X_1 and X_2 are independent random variables if and only if

$$P(a < X_1 \leq b, c < X_2 \leq d) = P(a < X_1 \leq b)P(c < X_2 \leq d)$$

for every $a < b$ and $c < d$, where a, b, c, d are constants.

Proof (continued). Now suppose

$$P(a < X_1 \leq b,\, c < X_2 \leq d) = P(a < X_1 \leq b)P(x < X_2 \leq d).$$

By Continuity of the Probability Function (Theorem 1.3.6), we have when $a \to -\infty$ and $c \to -\infty$ that $P(X_1 \leq b,\, x_2 \leq d) = P(X_1 \leq b)P(X_2 \leq d)$, or $F(b, d) = F_1(b)F_2(d)$. Since $b, d \in \mathbb{R}$ are arbitrary then $F(x_1, x_2) = F_1(x_1)F_2(x_2)$ for all $(x_1, x_2) \in \mathbb{R}^2$ and so by Theorem 2.4.2 X_1 and X_2 are independent, as claimed. □

Theorem 2.4.4

Theorem 2.4.4. Suppose X_1 and X_2 are independent and that $E[u(X_1)]$ and $E[v(X_2)]$ exists. Then

$$E[u(x_1)v(X_2)] = E[u(X_1)]E[v(X_2)].$$

Proof. We give the proof for continuous random variables and leave the discrete case as an exercise. Since X_1 and X_2 are hypothesized to be independent then $f(x_1, x_2) = f_1(x_1)f_2(x_2)$. So

$$E[u(X_1)v(X_2)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} u(x_1)v(x_2)f(x_1, x_2)\, dx_1\, dx_2$$

$$= \int_{-\infty}^{\infty} u(x_1)f_1(x_1)\, dx_1 \int_{-\infty}^{\infty} v(x_2)f_2(x_2)\, dx_2 = E[u(X_1)]E[v(X_2)],$$

as claimed. □

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Proof. We give the proof for continuous random variables and leave the discrete case as an exercise. Since X_1 and X_2 are hypothesized to be independent then $f(x_1, x_2) = f_1(x_1)f_2(x_2)$. So

$$E[u(X_1)v(X_2)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} u(x_1)v(x_2)f(x_1,x_2)\,dx_1\,dx_2$$

$$= \int_{-\infty}^{\infty} u(x_1)f_1(x_1)\,dx_1 \int_{-\infty}^{\infty} v(x_2)f_2(x_2)\,dx_2 = E[u(X_1)]E[v(X_2)],$$

as claimed. □

Theorem 2.4.5

Theorem 2.4.5. Suppose the joint moment generating function $M(t_1, t_2)$ exists for the random variables X_1 and X_2. Then X_1 and X_2 are independent if and only if $M(t_1, t_2) = M(t_1, 0)M(0, t_2)$; that is, the joint moment generating function is identically equal to the product of the marginal moment generating functions.

Proof. If X_1 and X_2 are independent then

$$\begin{aligned} M(t_1, t_2) &= E[e^{t_1X_1+t_2X_2}] = E[e^{t_1X_1}e^{t_2X_2}] \\ &= E[e^{t_1X_1}]E[e^{t_2X_2}] \text{ by Theorem 2.4.4} \\ &= M(t_1, 0)M(0, t_2), \end{aligned}$$

as claimed.

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Proof. If X_1 and X_2 are independent then

$$\begin{aligned} M(t_1, t_2) &= E[e^{t_1 X_1 + t_2 X_2}] = E[e^{t_1 X_1} e^{t_2 X_2}] \\ &= E[e^{t_1 X_1}] E[e^{t_2 X_2}] \text{ by Theorem 2.4.4} \\ &= M(t_1, 0) M(0, t_2), \end{aligned}$$

as claimed.

Now suppose $M(t_1, t_2) = M(t_1, 0)M(0, t_2)$. By Theorem 1.9.2, the moment generating function of a random variable is unique, so $M(t_1, 0) = \int_{-\infty}^{\infty} e^{t_1 x_1} f_1(x_1)\, dx_1$ and similarly $M(0, t_2) = \int_{-\infty}^{\infty} e^{t_2 x_2} f_2(x_2)\, dx_2$.

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Proof. If X_1 and X_2 are independent then

$$\begin{aligned} M(t_1, t_2) &= E[e^{t_1X_1+t_2X_2}] = E[e^{t_1X_1}e^{t_2X_2}] \\ &= E[e^{t_1X_1}]E[e^{t_2X_2}] \text{ by Theorem 2.4.4} \\ &= M(t_1, 0)M(0, t_2), \end{aligned}$$

as claimed.
Now suppose $M(t_1, t_2) = M(t_1, 0)M(0, t_2)$. By Theorem 1.9.2, the moment generating function of a random variable is unique, so $M(t_1, 0) = \int_{-\infty}^{\infty} e^{t_1x_1} f_1(x_1)\, dx_1$ and similarly $M(0, t_2) = \int_{-\infty}^{\infty} e^{t_2x_2} f_2(x_2)\, dx_2$.

Theorem 2.4.5 (continued)

Proof (continued). So, by hypothesis,

$$M(t_1, 0)M(0, t_2) = \int_{-\infty}^{\infty} e^{t_1 x_1} f_1(x_1)\, dx_1 \int_{-\infty}^{\infty} e^{t_2 x_2} f_2(x_2)\, dx_2$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{t_1 x_1 + t_2 x_2} f_1(x_1) f_2(x_2)\, dx_1\, dx_2 = M(t_1, t_2).$$

But also

$$M(t_1, t_2) = \int_{-\infty}^{\infty} e^{t_1 x_1 + t_2 x_2} f(x_1, x_2)\, dx_1\, dx_2.$$

So we have the moment generating function based on two probability density functions. By Theorem 1.9.2, if moment generating functions are equal then the cumulative distribution functions are equal and hence (by Note 1.7.A) their probability density functions are equal. Hence, $f(x_1, x_2) \equiv f_1(x_1) f_2(x_2)$ so that by Definition 2.4.1, X_1 and X_2 are independent, as claimed. □

Theorem 2.4.5 (continued)

Proof (continued). So, by hypothesis,

$$M(t_1, 0)M(0, t_2) = \int_{-\infty}^{\infty} e^{t_1 x_1} f_1(x_1)\, dx_1 \int_{-\infty}^{\infty} e^{t_2 x_2} f_2(x_2)\, dx_2$$

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But also

$$M(t_1, t_2) = \int_{-\infty}^{\infty} e^{t_1 x_1 + t_2 x_2} f(x_1, x_2)\, dx_1\, dx_2.$$

So we have the moment generating function based on two probability density functions. By Theorem 1.9.2, if moment generating functions are equal then the cumulative distribution functions are equal and hence (by Note 1.7.A) their probability density functions are equal. Hence, $f(x_1, x_2) \equiv f_1(x_1) f_2(x_2)$ so that by Definition 2.4.1, X_1 and X_2 are independent, as claimed. □

Exercise 2.4.6(a)

Exercise 2.4.6(a). If $f(x_1, x_2) = e^{-x_1-x_2}$ for $0 < x_1 < \infty$, $0 < x_2 < \infty$, and 0 elsewhere, is the joint probability density function of random variables X_1 and X_2, show that X_1 and X_2 are independent and that $M(t_1, t_2) = \dfrac{1}{(1-t_1)(1-t_2)}$ for $t_1 < 1$, $t_2 < 1$.

Solution. We have

$$M(t_1, t_2) = E[e^{t_1 X_1 + t_2 X_2}] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{t_1 x_1 + t_2 x_2} f(x_1, x_2)\, dx_1\, dx_2$$

$$= \int_0^{\infty} \int_0^{\infty} e^{t_1 x_1 + t_2 x_2} e^{-x_1 - x_2}\, dx_1\, dx_2 = \int_0^{\infty} \int_0^{\infty} e^{t_1 x_1 - x_1} e^{t_2 x_2 - x_2}\, dx_1\, dx_2$$

$$= \int_0^{\infty} e^{(t_1 - 1)x_1}\, dx_1 \int_0^{\infty} e^{(t_2 - 1)x_2}\, dx_2 = \cdots$$

Exercise 2.4.6(a)

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Solution. We have

$$M(t_1, t_2) = E[e^{t_1X_1+t_2X_2}] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{t_1x_1+t_2x_2} f(x_1, x_2)\, dx_1\, dx_2$$

$$= \int_0^{\infty}\int_0^{\infty} e^{t_1x_1+t_2x_2} e^{-x_1-x_2}\, dx_1\, dx_2 = \int_0^{\infty}\int_0^{\infty} e^{t_1x_1-x_1} e^{t_2x_2-x_2}\, dx_1\, dx_2$$

$$= \int_0^{\infty} e^{(t_1-1)x_1}\, dx_1 \int_0^{\infty} e^{(t_2-1)x_2}\, dx_2 = \cdots$$

Exercise 2.4.6(a) (continued 1)

Solution (continued).

$$\cdots = \left(\frac{1}{t_1 - 1} e^{(t_1-1)x_1} \Big|_{x_1=0}^{x_1=\infty} \right) \left(\frac{1}{t_2 - 1} e^{(t_2-1)x_2} \Big|_{x_2=0}^{x_2=\infty} \right)$$

$$= \left(0 - \frac{1}{t_1 - 1}\right)\left(0 \frac{1}{t_2 - 1}\right) \text{ if } t_1 < 1 \text{ and } t_2 < 1$$

$$= \frac{1}{(1-t_1)(1-t_2)} \text{ if } t_1 < 1 \text{ and } t_2 < 1, \text{ as claimed.}$$

We then have $M(t_1, 0) = -1/(1 - t_1)$ and $M(0, t_2) = -1/(1 - t_2)$ so that

$$M(t_1, t_2) = \frac{1}{(1-t_1)(1-t_2)} = \frac{-1}{1-t_1}\frac{-1}{1-t_2} = M(t_1, 0)M(0, t_2)$$

and so X_1 and X_2 are independent by Theorem 2.4.5. □

. . .

Exercise 2.4.6(a) (continued 1)

Solution (continued).

$$\cdots = \left(\frac{1}{t_1 - 1} e^{(t_1-1)x_1} \Big|_{x_1=0}^{x_1=\infty} \right) \left(\frac{1}{t_2 - 1} e^{(t_2-1)x_2} \Big|_{x_2=0}^{x_2=\infty} \right)$$

$$= \left(0 - \frac{1}{t_1 - 1} \right) \left(0 \frac{1}{t_2 - 1} \right) \text{ if } t_1 < 1 \text{ and } t_2 < 1$$

$$= \frac{1}{(1-t_1)(1-t_2)} \text{ if } t_1 < 1 \text{ and } t_2 < 1, \text{ as claimed.}$$

We then have $M(t_1, 0) = -1/(1 - t_1)$ and $M(0, t_2) = -1/(1 - t_2)$ so that

$$M(t_1, t_2) = \frac{1}{(1-t_1)(1-t_2)} = \frac{-1}{1-t_1} \frac{-1}{1-t_2} = M(t_1, 0)M(0, t_2)$$

and so X_1 and X_2 are independent by Theorem 2.4.5. □

. . .

Exercise 2.4.6(a) (continued 2)

Exercise 2.4.6(a). If $f(x_1, x_2) = e^{-x_1 - x_2}$ for $0 < x_1 < \infty$, $0 < x_2 < \infty$, and 0 elsewhere, is the joint probability density function of random variables X_1 and X_2, show that X_1 and X_2 are independent and that $M(t_1, t_2) = \dfrac{1}{(1 - t_1)(1 - t_2)}$ for $t_1 < 1$, $t_2 < 1$.

Note. We can easily show that X_1 and X_2 are independent using Theorem 2.4.1 (notice that the support of $f(x_1, x_2)$ is a "product space") with $g(x_1) = e^{-x_1}$ for $0 < x_1 < \infty$ and 0 elsewhere, and $h(x_2) = e^{-x_2}$ for $0 < x_2 < \infty$ and 0 elsewhere, so that $f(x_1, x_2) \equiv g(x_1)h(x_2)$.