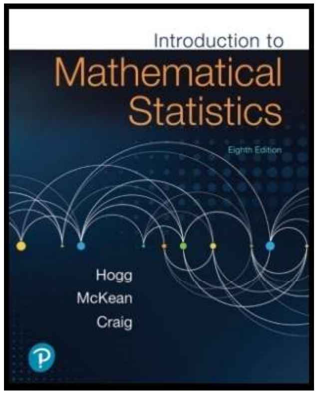
Mathematical Statistics 1

Chapter 2. Multivariate Distributions

2.6. Extension to Several Random Variables—Proofs of Theorems



Theorem 2.6.1

Theorem 2.6.1. Suppose $X_1, X_2, \ldots, X_n$ are n mutually independent random variables. Suppose the Moment generating function for x_i is $M_i(t)$ for $-j_1 < t < h_i$ where $h_i > 0$, for $i = 1, 2, \ldots, n$. Let $T = \sum_{i=1}^n k_i X_i$ where $k_1, k_2, \ldots, k_n$ are constants. Then T has the moment generating function given by
$M_T(i) = \prod_{i=1}^n M_i(k_i t)$ for $-\min_{1\le i\le n}\{h_i\} \le t \le \min_{1\le i\le n}\{h_i\}$.

Proof. Assume t is in the interval $(-\min_{1\le i\le n}\{h_i\}, \min_{1\le i\le n}\{h_i\})$. Then

$$\begin{aligned} M_T(t) &= E\left[\exp\left(\sum_{i=1}^n tk_iX_i\right)\right] = E\left[\prod_{i=1}^n e^{ik_iX_i}\right] \\ &= \prod_{i=1}^n E\left[e^{tk_iX_i}\right] \text{ by the mutual independence} \\ &= \prod_{i=1}^n M_i(k_it). \quad \square \end{aligned}$$

Theorem 2.6.2

Theorem 2.6.2. Let $\mathbf{V}$ and $\mathbf{W}$ be $m \times n$ matrices of random variables, let $\mathbf{A}$ and $\mathbf{C}$ be $k \times m$ matrices of constants, and let $\mathbf{B}$ be an $n \times \ell$ matrix of constants. Then $E[\mathbf{AV} + \mathbf{CW}] = \mathbf{A}E[\mathbf{V}] + \mathbf{C}E[\mathbf{W}]$ and
$E[\mathbf{AWB}] = \mathbf{A}E[\mathbf{E}]\mathbf{B}$; that is, E is a linear operator on matrices of random variables.

Proof. Since E is a linear operator on random variable by Theorem 2.1.1, then the (i,j) component of $E[\mathbf{AV} + \mathbf{CW}]$ is

$$E\left[\sum_{s=1}^m a_{is}V_{sj} + \sum_{s=1}^m c_{is}W_{sj}\right] = \sum_{s=1}^m a_{is}E[V_{sj}] + \sum_{s=1}^m c_{is}E[W_{sj}]$$

and the first claim holds.

Next, the (i,p) entry of $\mathbf{AW}$ (an $k \times m$ matrix) is $\sum_{s=1}^m a_{is}W_{sp}$ and the (i,j) entry of $\mathbf{AWB}$ (an $k \times \ell$ matrix) is $\sum_{p=1}^n (\sum_{s=1}^m a_{is}W_{sp})\, b_{pj}$.

Theorem 2.6.2 (continued)

Proof (continued). Since E is a linear operator on random variables by Theorem 2.1.1, then the (i,j) entry of $E[\mathbf{AWB}]$ is

$$E\left[\sum_{p=1}^n \left(\sum_{s=1}^m a_{is}W_{sp}\right) b_{pj}\right] = \sum_{p=1}^m E\left[\sum_{s=1}^m a_{is}W_{sp}b_{pj}\right]$$

$$= \sum_{p=1}^m E\left[\sum_{s=1}^m a_{is}W_{sp}\right] b_{pj} = \sum_{p=1}^m \left(\sum_{s=1}^m a_{is}E[W_{sp}]\right) b_{pj},$$

and this is the (i,j) entry of $\mathbf{A}E[\mathbf{W}]\mathbf{B}$, so the second claim holds. $\square$

Theorem 2.6.3

Theorem 2.6.3. Let $\mathbf{X} = (X_1, X_2, \ldots, X_n)' = (X_1, X_2, \ldots, X_n)^T$ be an n-dimensional random vector, such that $\sigma_i^2 = \sigma_{ii} = \text{Var}(X_i) < \infty$. Let $\mathbf{A}$ be an $m \times n$ matrix of constants. Then $\text{Cov}(\mathbf{X}) = E[\mathbf{X}\mathbf{X}'] = \boldsymbol{\mu}\boldsymbol{\mu}'$ and $\text{Cov}(\mathbf{A}\mathbf{X}) = \mathbf{A}\text{Cov}(\mathbf{X})\mathbf{A}'$.

Proof. First,

$$\begin{aligned}
\text{Cov}(\mathbf{X}) &= E[(\mathbf{X}-\boldsymbol{\mu})(\mathbf{X}-\boldsymbol{\mu})'] \text{ by definition} \\
&= E[\mathbf{X}\mathbf{X}' - \boldsymbol{\mu}\mathbf{X}' - \mathbf{X}\boldsymbol{\mu}' + \boldsymbol{\mu}\boldsymbol{\mu}'] \\
&= E[\mathbf{X}\mathbf{X}'] - \boldsymbol{\mu}E[\mathbf{X}'] = E[\mathbf{X}]\boldsymbol{\mu}' + E[\boldsymbol{\mu}\boldsymbol{\mu}'] \text{ by Theorem 2.6.2} \\
&= E[\mathbf{X}\mathbf{X}'] - \boldsymbol{\mu}\boldsymbol{\mu}' - \boldsymbol{\mu}\boldsymbol{\mu}' + \boldsymbol{\mu}\boldsymbol{\mu}' \\
&= E[\mathbf{X}\mathbf{X}'] - \boldsymbol{\mu}\boldsymbol{\mu}',
\end{aligned}$$

as claimed.

Theorem 2.6.3 (continued)

Theorem 2.6.3. Let $\mathbf{X} = (X_1, X_2, \ldots, X_n)' = (X_1, X_2, \ldots, X_n)^T$ be an n-dimensional random vector, such that $\sigma_i^2 = \sigma_{ii} = \text{Var}(X_i) < \infty$. Let $\mathbf{A}$ be an $m \times n$ matrix of constants. Then $\text{Cov}(\mathbf{X}) = E[\mathbf{X}\mathbf{X}'] = \boldsymbol{\mu}\boldsymbol{\mu}'$ and $\text{Cov}(\mathbf{A}\mathbf{X}) = \mathbf{A}\text{Cov}(\mathbf{X})\mathbf{A}'$.

Proof (continued). Next, by Theorem 2.6.2, $E[\mathbf{A}\mathbf{X}] = \mathbf{A}E[\mathbf{X}] = \mathbf{A}\boldsymbol{\mu}$ and

$$\begin{aligned}
\text{Cov}(\mathbf{A}\mathbf{X}) &= E[(\mathbf{A}\mathbf{A} - \mathbf{A}\boldsymbol{\mu})(\mathbf{A}\mathbf{X} - \mathbf{A}\boldsymbol{\mu})'] \text{ by definition} \\
&= E[(\mathbf{A}\mathbf{A} - \mathbf{A}\boldsymbol{\mu})(\mathbf{X}'\mathbf{A}' - \boldsymbol{\mu}'\mathbf{A}')] \\
&\quad \text{since } (\mathbf{A}\mathbf{B})' = (\mathbf{A}\mathbf{B})^T = \mathbf{B}^T\mathbf{A}^T = \mathbf{B}'\mathbf{A}' \\
&= E[\mathbf{A}\mathbf{X}\mathbf{X}'\mathbf{A}' - \mathbf{A}\boldsymbol{\mu}\mathbf{X}'\mathbf{A}' - \mathbf{A}\mathbf{X}\boldsymbol{\mu}'\mathbf{A}' + \mathbf{A}\boldsymbol{\mu}\boldsymbol{\mu}'\mathbf{A}'] \\
&= \text{Cov}(\mathbf{X})\mathbf{A}' \text{ by the first result.}
\end{aligned}$$

□

Corollary 2.6.A

Corollary 2.6.A. All variance-covariance matrices are positive semi-definite.

Proof. Let $\mathbf{X}$ be a random (column) vector of n random variables and let $\mathbf{a}$ be a constant $n \times 1$ vector. Then $Y = \mathbf{a}'\mathbf{X}$ is a random variable (a linear combination of the components of $\mathbf{X}$) and so has a nonnegative variance. That is,

$$\begin{aligned}
0 &\le \text{Var}(Y) = \text{Var}(\mathbf{a}'\mathbf{X}) \\
&= E[(\mathbf{a}'\mathbf{X} - E[\mathbf{a}'\mathbf{X}])^2] \text{ by Definition 1.9.2} \\
&= \text{Cov}(\mathbf{a}'\mathbf{X}) \text{ since } \text{Cov}(Y) = \text{Var}(Y) \text{ for a single random variable} \\
&= \mathbf{a}'\text{Cov}(\mathbf{X})\mathbf{a} \text{ by Theorem 2.6.3.}
\end{aligned}$$

So $\text{Cov}(\mathbf{X})$ is a positive semi-definite matrix, as claimed. □