Mathematical Statistics 1

Chapter 2. Multivariate Distributions
2.8. Linear Combinations of Random Variables—Proofs of Theorems

Theorem 2.8.1

Let $X_1, X_2, \ldots, X_n$ be random variables and define $T = \sum_{i=1}^{n} a_i X_i$. Suppose $E[X_i] = \mu_i$ for $i = 1, 2, \ldots, n$. Then $E[T] = \sum_{i=1}^{n} a_i \mu_i$.

Proof. By Theorem 2.1.1 (and induction) $E$ is linear so

$$E[T] = E \left[ \sum_{i=1}^{n} a_i X_i \right] = \sum_{i=1}^{n} E[a_i X_i] = \sum_{i=1}^{n} a_i E[X_i] = \sum_{i=1}^{n} a_i \mu_i,$$

as claimed.

Theorem 2.8.2

Let $X_1, X_2, \ldots, X_n, Y_1, Y_2, \ldots, Y_m$ be random variables and define $T = \sum_{i=1}^{n} a_i X_i$ and $W = \sum_{j=1}^{m} b_j Y_j$. If $E[X_i^2] < \infty$ and $E[Y_j^2] < \infty$ for $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, m$, then

$$\text{Cov}(T, W) = \sum_{i=1}^{n} \sum_{j=1}^{m} a_i b_j \text{Cov}(X_i, Y_j).$$

Proof. We have by the definition of covariance (Definition 2.5.1) that

$$\text{Cov}(T, W) = E[(T - \mu_T)(W - \mu_W)] = E[T] = \mu_T$$

is the mean of $T$ and $E[W] = \mu_W$ is the mean of $W$

$$= E \left[ \left( \sum_{i=1}^{n} a_i X_i - \sum_{i=1}^{n} a_i E[X_i] \right) \left( \sum_{j=1}^{m} b_j Y_j - \sum_{j=1}^{m} b_j E[Y_j] \right) \right]$$

$$= E \left[ \left( \sum_{i=1}^{n} (a_i X_i - a_i E[X_i]) \right) \left( \sum_{j=1}^{m} (b_j Y_j - b_j E[Y_j]) \right) \right].$$

Theorem 2.8.2 (continued 1)

Proof (continued).

$$\text{Cov}(T, W) = E \left[ \sum_{i=1}^{n} \sum_{j=1}^{m} (a_i X_i - a_i E[X_i])(b_j Y_j - b_j E[Y_j]) \right]$$

$$= \sum_{i=1}^{n} \sum_{j=1}^{m} E[(a_i X_i - a_i E[X_i])(b_j Y_j - b_j E[Y_j])]
$$

since $E$ is linear by Theorem 2.1.1

$$= \sum_{i=1}^{n} \sum_{j=1}^{m} E[a_i b_j X_i Y_j - a_i b_j E[X_i] Y_j]
- a_i b_j E[X_i] E[Y_j]
+ a_i b_j E[X_i] E[Y_j]$$

$$= \sum_{i=1}^{n} \sum_{j=1}^{m} (a_i b_j E[X_i] Y_j - a_i b_j E[X_i] E[Y_j])$$

by Theorem 2.1.1
Theorem 2.8.2 (continued 2)

Proof (continued). . . .

\[ \text{Cov}(T, W) = \sum_{i=1}^{n} \sum_{j=1}^{m} a_i b_j (E[X_i Y_j] - 2E[X_i]E[Y_j] + E[X_i]E[Y_j]) \]

\[ = \sum_{i=1}^{n} \sum_{j=1}^{m} a_i b_j E[(X_i - E[X_i])(Y_j - E[Y_j])] \]

by Theorem 2.1.1

\[ = \sum_{i=1}^{n} \sum_{j=1}^{m} a_i b_j \text{Cov}(X_i, Y_j) \]

by Definition 2.5.1,

as claimed. \(\square\)

Corollary 2.8.1 (continued)

Corollary 2.8.1. Let \(X_1, X_2, \ldots, X_n\) be random variables and define

\[ T = \sum_{i=1}^{n} a_i X_i. \]

Provided \(E[X_i^2] < \infty\) for \(i = 1, 2, \ldots, n\), then

\[ \text{Var}(T) = \sum_{i=1}^{n} a_i^2 \text{Var}(X_i) + s \sum_{i < j} a_i a_j \text{Cov}(X_i, X_j). \]

Proof. By the definition of variance (Definition 1.9.2) we have

\[ \text{Var}(T) = E[(T - E[T])^2] = E[(T - E[T])(T - E[T])] \]

by Definition 2.5.1

\[ = \sum_{i=1}^{n} \sum_{j=1}^{m} a_i a_j \text{Cov}(X_i, X_j) + \sum_{i=1}^{n} \sum_{j=1, j \neq i}^{m} a_i a_j \text{Cov}(X_i, X_j) \]

\[ = \sum_{i=1}^{n} a_i^2 \text{Var}(X_i) + 2 \sum_{i=1}^{n} \sum_{j=i+1}^{m} a_i a_j \text{Cov}(X_i, X_j) \]

since \(\text{Var}(X_i) = \text{Cov}(X_i, X_i)\) and \(\text{Cov}(X_i, X_j) > \text{Cov}(X_j, X_i)\)

Corollary 2.8.2

Corollary 2.8.2. Let \(X_1, X_2, \ldots, X_n\) be independent random variables and define

\[ T = \sum_{i=1}^{n} a_i X_i. \]

With \(\text{Var}(X_i) = \sigma_i^2\) for \(i = 1, 2, \ldots, n\) we have

\[ \text{Var}(T) = \sum_{i=1}^{n} a_i^2 \sigma_i^2. \]

Proof. Since \(X_i\) and \(X_j\) are independent for \(i \neq j\) then by Theorem 2.5.2

\[ \text{Cov}(X_i, X_j) = 0 \] for \(i \neq j\). So by Corollary 2.8.1,

\[ \text{Var}(T) = \text{Cor}(T, T) = \sum_{i=1}^{n} a_i^2 \text{Var}(X_i) + 2 \sum_{i < j} a_i a_j \text{Cov}(X_i, X_j) \]

\[ = \sum_{i=1}^{n} a_i^2 \sigma_i^2 + 0 = \sum_{i=1}^{n} a_i^2 \sigma_i^2, \]

as claimed. \(\square\)
Theorem 2.8.A

Theorem 2.8.A. Let \( X_1, X_2, \ldots, X_n \) be independent and identically distributed random variables with common mean \( \mu \) and variance \( \sigma^2 \). We have

\[
E[X] = \mu, \quad \text{Var}(X) = \frac{\sigma^2}{n}, \quad S^2 = \frac{\sum_{i=1}^{n} X_i^2 - n\bar{X}^2}{n-1}, \quad \text{and} \quad E[S^2] = \sigma^2.
\]

Proof. We have \( \bar{X} = \frac{\sum_{i=1}^{n} X_i}{n} = \frac{\sum_{i=1}^{n} X_i}{n} \) by definition, so by Theorem 2.8.1

\[
E[\bar{X}] = \sum_{i=1}^{n} \frac{1}{n} E[X_i] = \sum_{i=1}^{n} \frac{\mu}{n} = \mu.
\]

By Corollary 2.8.2, \( \text{Var}(\bar{X}) = \sum_{i=1}^{n} \frac{1}{n^2} \sigma^2 = \frac{\sigma^2}{n} \). The proof that

\[
S^2 = \frac{\sum_{i=1}^{n} X_i^2 - n\bar{X}^2}{n-1}
\]

is to be given in Exercise 2.8.1.

Proof (continued). Finally,

\[
E[S^2] = E\left[ \frac{\sum_{i=1}^{n} X_i^2 - n\bar{X}^2}{n-1} \right]
\]

\[
= \frac{1}{n-1} \sum_{i=1}^{n} E[X_i^2] - nE[\bar{X}^2]
\]

by Theorem 2.1.1

\[
= \frac{n(\sigma^2 - \mu^2) - nE[\bar{X}^2]}{n-1}
\]

since \( E[X^2] = \sigma^2 + \mu^2 \) by Note 1.9.A

\[
= \frac{n(\sigma^2 + \mu^2 - n(\sigma^2/n + \mu^2))}{n-1}
\]

since \( E(\bar{X}^2) = \text{Var}(\bar{X}) + E(\bar{X})^2 \)

\[
= \frac{\sigma^2}{n} + \mu^2
\]

\[
= \frac{(n-1)\sigma^2}{n-1} = \sigma^2,
\]

as claimed.