Mathematical Statistics 1

Chapter 2. Multivariate Distributions

2.8. Linear Combinations of Random Variables—Proofs of Theorems

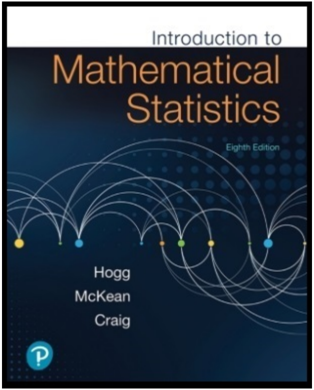

Table of contents

1. Theorem 2.8.1
2. Theorem 2.8.2
3. Corollary 2.8.1
4. Corollary 2.8.2
5. Theorem 2.8.A

Theorem 2.8.1

Theorem 2.8.1. Let $X_1, X_2, \ldots, X_n$ be random variables and define $T = \sum_{i=1}^n a_i X_i$. Suppose $E[X_i] = \mu_i$ for $i = 1, 2, \ldots, n$. Then $E[T] = \sum_{i=1}^n a_i \mu_i$.

Proof. By Theorem 2.1.1 (and induction) E is linear so

$$E[T] = E\left[\sum_{i=1}^n a_i X_i\right] = \sum_{i=1}^n E[a_i X_i] = \sum_{i=1}^n a_i E[X_i] = \sum_{i=1}^n a_i \mu_i,$$

as claimed. □

Theorem 2.8.1

Theorem 2.8.1. Let $X_1, X_2, \ldots, X_n$ be random variables and define $T = \sum_{i=1}^n a_i X_i$. Suppose $E[X_i] = \mu_i$ for $i = 1, 2, \ldots, n$. Then $E[T] = \sum_{i=1}^n a_i \mu_i$.

Proof. By Theorem 2.1.1 (and induction) E is linear so

$$E[T] = E\left[\sum_{i=1}^n a_i X_i\right] = \sum_{i=1}^n E[a_i X_i] = \sum_{i=1}^n a_i E[X_i] = \sum_{i=1}^n a_i \mu_i,$$

as claimed. $\square$

Theorem 2.8.2

Theorem 2.8.2. Let $X_1, X_2, \ldots, X_n, Y_1, Y_2, \ldots, Y_m$ be random variables and define $T = \sum_{i=1}^n a_i X_i$ and $W = \sum_{j=1}^m b_j Y_j$. If $E[X_i^2] < \infty$ and $E[Y_j^2] < \infty$ for $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, m$, then

$$\mathrm{Cov}(T, W) = \sum_{i=1}^n \sum_{j=1}^m a_i b_j \,\mathrm{Cov}(X_i, Y_j).$$

Proof. We have by the definition of covariance (Definition 2.5.1) that

$$\begin{aligned}
\mathrm{Cov}(T, W) &= E[(T - \mu_T)(W - \mu_W)] \text{ where } E[T] = \mu_T \text{ is the} \\
&\quad \text{mean of } T \text{ and } E[W] = \mu_W \text{ is the mean of } W \\
&= E\left[\left(\sum_{i=1}^n a_i X_i = \sum_{i=1}^n a_i E[X_i]\right)\left(\sum_{j=1}^m b_j Y_j - \sum_{j=1}^m b_j E[Y_j]\right)\right] \\
&= E\left[\left(\sum_{i=1}^n (a_i X_i - a_i E[X_i])\right)\left(\sum_{j=1}^m (b_j Y_j - b_j E[Y_j])\right)\right]
\end{aligned}$$

Theorem 2.8.2

Theorem 2.8.2. Let $X_1, X_2, \ldots, X_n$, Y_1, $Y_2, \ldots, Y_m$ be random variables and define $T = \sum_{i=1}^n a_i X_i$ and $W = \sum_{j=1}^m b_j Y_j$. If $E[X_i^2] < \infty$ and $E[Y_j^2] < \infty$ for $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, m$, then

$$\mathrm{Cov}(T, W) = \sum_{i=1}^n \sum_{j=1}^m a_i b_j \, \mathrm{Cov}(X_i, Y_j).$$

Proof. We have by the definition of covariance (Definition 2.5.1) that

$$\begin{aligned}
\mathrm{Cov}(T, W) &= E[(T - \mu_T)(W - \mu_W)] \text{ where } E[T] = \mu_T \text{ is the} \\
&\quad \text{mean of } T \text{ and } E[W] = \mu_W \text{ is the mean of } W \\
&= E\left[\left(\sum_{i=1}^n a_i X_i = \sum_{i=1}^n a_i E[X_i]\right)\left(\sum_{j=1}^m b_j Y_j - \sum_{j=1}^m b_j E[Y_j]\right)\right] \\
&= E\left[\left(\sum_{i=1}^n (a_i X_i - a_i E[X_i])\right)\left(\sum_{j=1}^m (b_j Y_j - b_j E[Y_j])\right)\right]
\end{aligned}$$

Theorem 2.8.2 (continued 1)

Proof (continued).

$$
\begin{aligned}
\mathrm{Cov}(T, W) &= E\left[\sum_{i=1}^{n}\sum_{j=1}^{m}(a_iX_i - a_iE[X_i])(b_jY_j - b_jE[Y_j])\right] \\
&= \sum_{i=1}^{n}\sum_{j=1}^{m}E[(a_iX_i - a_iE[X_i])(b_jY_j - b_jE[Y_j])] \\
&\quad \text{since } E \text{ is linear by Theorem 2.1.1} \\
&= \sum_{i=1}^{n}\sum_{j=1}^{m}E[a_ib_jX_iY_j - a_ib_jE[X_i]Y_j \\
&\quad -a_ib_jE[X_i]Y_j + a_ib_jE[X_i]E[Y_j]) \\
&= \sum_{i=1}^{n}\sum_{j=1}^{m}(a_ib_jE[X_iY_j] - a_ib_jE[X_i]E[Y_j] \\
&\quad -a_ib_jE[X_i]E[Y_j] + a_ib_jE[X_i]E[Y_j]) \text{ by Theorem 2.1.1}
\end{aligned}
$$

Theorem 2.8.2 (continued 2)

Proof (continued). ...

$$\begin{aligned}
\mathrm{Cov}(T,W) &= \sum_{i=1}^{n}\sum_{j=1}^{m} a_i b_j (E[X_i Y_j] - 2E[X_i]E[Y_j] + E[X_i]E[Y_j]) \\
&= \sum_{i=1}^{n}\sum_{j=1}^{m} a_i b_j E[(X_i - E[X_i])(Y_j - E[Y_j])] \text{ by Theorem 2.1.1} \\
&= \sum_{i=1}^{n}\sum_{j=1}^{m} a_i b_j \,\mathrm{Cov}(X_i, Y_j) \text{ by Definition 2.5.1,}
\end{aligned}$$

as claimed. □

Corollary 2.8.1

Corollary 2.8.1. Let $X_1, X_2, \ldots, X_n$ be random variables and define $T = \sum_{i=1}^{n} a_i X_i$. Provided $E[X_i^2] < \infty$ for $i = 1, 2, \ldots, n$, then

$$\operatorname{Var}(T) = \sum_{i=1}^{n} a_i^2 \operatorname{Var}(X_i) + s \sum_{i<j} a_i a_j \operatorname{Cov}(X_i, X_j).$$

Proof. By the definition of variance (Definition 1.9.2) we have

$$\begin{aligned}
\operatorname{Var}(T) &= E[(T - E[T])^2) = E[(T - E[T])(T - E[T])] \\
&= \operatorname{Cov}(T, T) \text{ by Definition 2.5.1} \\
&= \sum_{i=1}^{n} \sum_{j=1}^{m} a_i a_j \operatorname{Cov}(X_i, X_i) + \sum_{i=1}^{n} \sum_{j=1, j \neq i}^{m} a_i a_j \operatorname{Cov}(X_i, X_j) \\
&= \sum_{i=1}^{n} a_i^2 \operatorname{Var}(X_i) + 2 \sum_{i=1}^{n} \sum_{j=i+1}^{m} a_i a_j \operatorname{Cov}(X_i, X_j) \\
&\quad \text{since } \operatorname{Var}(X_i) = \operatorname{Cov}(X_i, X_i) \text{ and } \operatorname{Cov}(X_i, X_j) > \operatorname{Cov}(X_j, X_i)
\end{aligned}$$

Corollary 2.8.1

Corollary 2.8.1. Let $X_1, X_2, \ldots, X_n$ be random variables and define $T = \sum_{i=1}^n a_i X_i$. Provided $E[X_i^2] < \infty$ for $i = 1, 2, \ldots, n$, then

$$\mathrm{Var}(T) = \sum_{i=1}^n a_i^2 \,\mathrm{Var}(X_i) + s \sum_{i<j} a_i a_j \,\mathrm{Cov}(X_i, X_j).$$

Proof. By the definition of variance (Definition 1.9.2) we have

$$\begin{aligned}
\mathrm{Var}(T) &= E[(T - E[T])^2) = E[(T - E[T])(T - E[T])] \\
&= \mathrm{Cov}(T, T) \text{ by Definition 2.5.1} \\
&= \sum_{i=1}^n \sum_{j=1}^m a_i a_j \,\mathrm{Cov}(X_i, X_i) + \sum_{i=1}^n \sum_{j=1, j\neq i}^m a_i a_j \,\mathrm{Cov}(X_i, X_j) \\
&= \sum_{i=1}^n a_i^2 \,\mathrm{Var}(X_i) + 2 \sum_{i=1}^n \sum_{j=i+1}^m a_i a_j \,\mathrm{Cov}(X_i, X_j) \\
&\quad \text{since } \mathrm{Var}(X_i) = \mathrm{Cov}(X_i, X_i) \text{ and } \mathrm{Cov}(X_i, X_j) > \mathrm{Cov}(X_j, X_i)
\end{aligned}$$

Corollary 2.8.1 (continued)

Corollary 2.8.1. Let $X_1, X_2, \ldots, X_n$ be random variables and define $T = \sum_{i=1}^n a_i X_i$. Provided $E[X_i^2] < \infty$ for $i = 1, 2, \ldots, n$, then

$$\mathrm{Var}(T) = \sum_{i=1}^n a_i^2 \,\mathrm{Var}(X_i) + s \sum_{i<j} a_i a_j \,\mathrm{Cov}(X_i, X_j).$$

Proof (continued). ...

$$\begin{aligned} \mathrm{Var}(T) &= \sum_{i=1}^n a_i^2 \,\mathrm{Var}(X_i) + 2 \sum_{i=1}^n \sum_{j=i+1}^m a_i a_j \,\mathrm{Cov}(X_i, X_j) \\ &= \sum_{i=1}^n a_i^2 \,\mathrm{Var}(X_i) + 2 \sum_{i<j} a_i a_j \,\mathrm{Cov}(X_i, X_j), \end{aligned}$$

as claimed. □

Corollary 2.8.2

Corollary 2.8.2. Let $X_1, X_2, \ldots, X_n$ be independent random variables and define $T = \sum_{i=1}^n a_i X_i$. With $\text{Var}(X_i) = \sigma_i^2$ for $i = 1, 2, \ldots, n$ we have $\text{Var}(T) = \sum_{i=1}^n a_i^2 \sigma_i^2$.

Proof. Since X_i and X_j are independent for $i \neq j$ then by Theorem 2.5.2 $\text{Cov}(X_i, X_j) = 0$ for $i \neq j$. So by Corollary 2.8.1,

$$\text{Var}(T) = \text{Cor}(T, T) = \sum_{i=1}^n a_i^2 \text{Var}(X_i) + 2 \sum_{i<j} a_i a_j \, \text{Cov}(X_i, X_j)$$

$$= \sum_{i=1}^n a_i \sigma_i^2 + 0 = \sum_{i=1}^n a_i \sigma_i^2,$$

as claimed. □

Corollary 2.8.2

Corollary 2.8.2. Let $X_1, X_2, \ldots, X_n$ be independent random variables and define $T = \sum_{i=1}^n a_i X_i$. With $\mathrm{Var}(X_i) = \sigma_i^2$ for $i = 1, 2, \ldots, n$ we have $\mathrm{Var}(T) = \sum_{i=1}^n a_i^2 \sigma_i^2$.

Proof. Since X_i and X_j are independent for $i \neq j$ then by Theorem 2.5.2 $\mathrm{Cov}(X_i, X_j) = 0$ for $i \neq j$. So by Corollary 2.8.1,

$$\mathrm{Var}(T) = \mathrm{Cor}(T, T) = \sum_{i=1}^n a_i^2 \mathrm{Var}(X_i) + 2 \sum_{i<j} a_i a_j \,\mathrm{Cov}(X_i, X_j)$$

$$= \sum_{i=1}^n a_i \sigma_i^2 + 0 = \sum_{i=1}^n a_i \sigma_i^2,$$

as claimed. $\square$

Theorem 2.8.A

Theorem 2.8.A. Let $X_1, X_2, \ldots, X_n$ be independent and identically distributed random variables with common mean μ and variance σ^2. We have

$$E[\overline{X}] = \mu,\ \mathrm{Var}(\overline{X}) = \frac{\sigma^2}{n},\ S^2 = \frac{\sum_{i=1}^n X_i^2 - n\overline{X}^2}{n-1},\ \text{and } E[S^2] = \sigma^2.$$

Proof. We have $\overline{X} = \frac{\sum_{i=1}^n X_i}{n} = \sum_{i=1}^n \frac{X_i}{n}$ by definition, so by Theorem 2.8.1

$$E[\overline{X}] = \sum_{i=1}^n \frac{1}{n} E[X_i] = \sum_{i=1}^n \frac{\mu}{n} = \mu.$$

Theorem 2.8.A

Theorem 2.8.A. Let $X_1, X_2, \ldots, X_n$ be independent and identically distributed random variables with common mean μ and variance σ^2. We have

$$E[\overline{X}] = \mu, \ \mathrm{Var}(\overline{X}) = \frac{\sigma^2}{n}, \ S^2 = \frac{\sum_{i=1}^n X_i^2 - n\overline{X}^2}{n-1}, \text{ and } E[S^2] = \sigma^2.$$

Proof. We have $\overline{X} = \frac{\sum_{i=1}^n X_i}{n} = \sum_{i=1}^n \frac{X_i}{n}$ by definition, so by Theorem 2.8.1

$$E[\overline{X}] = \sum_{i=1}^n \frac{1}{n} E[X_i] = \sum_{i=1}^n \frac{\mu}{n} = \mu.$$

By Corollary 2.8.2, $\mathrm{Var}(\overline{X}) = \sum_{i=1}^n \frac{1}{n^2}\sigma^2 = \frac{\sigma^2}{n}$.

Theorem 2.8.A

Theorem 2.8.A. Let $X_1, X_2, \ldots, X_n$ be independent and identically distributed random variables with common mean μ and variance σ^2. We have

$$E[\overline{X}] = \mu,\ \mathrm{Var}(\overline{X}) = \frac{\sigma^2}{n},\ S^2 = \frac{\sum_{i=1}^n X_i^2 - n\overline{X}^2}{n-1},\ \text{and } E[S^2] = \sigma^2.$$

Proof. We have $\overline{X} = \frac{\sum_{i=1}^n X_i}{n} = \sum_{i=1}^n \frac{X_i}{n}$ by definition, so by Theorem 2.8.1

$$E[\overline{X}] = \sum_{i=1}^n \frac{1}{n} E[X_i] = \sum_{i=1}^n \frac{\mu}{n} = \mu.$$

By Corollary 2.8.2, $\mathrm{Var}(\overline{X}) = \sum_{i=1}^n \frac{1}{n^2}\sigma^2 = \frac{\sigma^2}{n}$. The proof that $S^2 = \frac{\sum_{i=1}^n X_i^2 - n\overline{X}^2}{n-1}$ is to be given in Exercise 2.8.1.

Theorem 2.8.A

Theorem 2.8.A. Let $X_1, X_2, \ldots, X_n$ be independent and identically distributed random variables with common mean μ and variance σ^2. We have

$$E[\overline{X}] = \mu,\ \mathrm{Var}(\overline{X}) = \frac{\sigma^2}{n},\ S^2 = \frac{\sum_{i=1}^n X_i^2 - n\overline{X}^2}{n-1},\ \text{and } E[S^2] = \sigma^2.$$

Proof. We have $\overline{X} = \frac{\sum_{i=1}^n X_i}{n} = \sum_{i=1}^n \frac{X_i}{n}$ by definition, so by Theorem 2.8.1

$$E[\overline{X}] = \sum_{i=1}^n \frac{1}{n} E[X_i] = \sum_{i=1}^n \frac{\mu}{n} = \mu.$$

By Corollary 2.8.2, $\mathrm{Var}(\overline{X}) = \sum_{i=1}^n \frac{1}{n^2}\sigma^2 = \frac{\sigma^2}{n}$. The proof that $S^2 = \frac{\sum_{i=1}^n X_i^2 - n\overline{X}^2}{n-1}$ is to be given in Exercise 2.8.1.

Theorem 2.8.A (continued)

Proof (continued). Finally,

$$
\begin{aligned}
E[S^2] &= E\left[\frac{\sum_{i=1}^n X_i^2 - n\overline{X}^2}{n-1}\right] \\
&= \frac{\sum_{i=1}^n E[X_i^2] - nE[\overline{X}^2]}{n-1} \text{ by Theorem 2.1.1} \\
&= \frac{n(\sigma^2 - \mu^2) - nE[\overline{X}^2]}{n-1} \text{ since } E[X^2] = \sigma^2 + \mu^2 \text{ by Note 1.9.A} \\
&= \frac{n(\sigma^2 + \mu^2) - n(\sigma^2/n + \mu^2)}{n-1} \text{ since } E(\overline{X}^2) = \mathrm{Var}(\overline{X})^2 + E(\overline{X})^2 \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad = \frac{\sigma^2}{n} + \mu^2 \\
&= \frac{(n-1)\sigma^2}{n-1} = \sigma^2,
\end{aligned}
$$

as claimed. □