Chapter 2. Multivariate Distributions
2.8. Linear Combinations of Random Variables—Proofs of Theorems
Theorem 2.8.1. Let $X_1, X_2, \ldots, X_n$ be random variables and define $T = \sum_{i=1}^{n} a_i X_i$. Suppose $E[X_i] = \mu_i$ for $i = 1, 2, \ldots, n$. Then $E[T] = \sum_{i=1}^{n} a_i \mu_i$.

Proof. By Theorem 2.1.1 (and induction) $E$ is linear so

$$E[T] = E \left[ \sum_{i=1}^{n} a_i X_i \right] = \sum_{i=1}^{n} E[a_i X_i] = \sum_{i=1}^{n} a_i E[X_i] = \sum_{i=1}^{n} a_i \mu_i,$$

as claimed.
Theorem 2.8.1

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as claimed.
**Theorem 2.8.2.** Let $X_1, X_2, \ldots, X_n, Y_1, Y_2, \ldots, Y_m$ be random variables and define $T = \sum_{i=1}^{n} a_i X_i$ and $W = \sum_{j=1}^{m} b_j Y_j$. If $E[X_i^2] < \infty$ and $E[Y_j^2] < \infty$ for $i = 1, 2, \ldots, n$ and $j = 1, 2, \ldots, m$, then

$$\text{Cov}(T, W) = \sum_{i=1}^{n} \sum_{j=1}^{m} a_i b_j \text{Cov}(X_i, Y_j).$$

**Proof.** We have by the definition of covariance (Definition 2.5.1) that

$$\text{Cov}(T, W) = E[(T - \mu_T)(W - \mu_W)]$$

where $E[T] = \mu_T$ is the mean of $T$ and $E[W] = \mu_W$ is the mean of $W$

$$= E \left[ \left( \sum_{i=1}^{n} a_i X_i - \sum_{i=1}^{n} a_i E[X_i] \right) \left( \sum_{j=1}^{m} b_j Y_j - \sum_{j=1}^{m} b_j E[Y_j] \right) \right]$$

$$= E \left[ \left( \sum_{i=1}^{n} (a_i X_i - a_i E[X_i]) \right) \left( \sum_{j=1}^{m} (b_j Y_j - b_j E[Y_j]) \right) \right]$$
Theorem 2.8.2

**Theorem 2.8.2.** Let \( X_1, X_2, \ldots, X_n, Y_1, Y_2, \ldots, Y_m \) be random variables and define \( T = \sum_{i=1}^{n} a_i X_i \) and \( W = \sum_{j=1}^{m} b_j Y_j \). If \( E[X_i^2] < \infty \) and \( E[Y_j^2] < \infty \) for \( i = 1, 2, \ldots, n \) and \( j = 1, 2, \ldots, m \), then

\[
\text{Cov}(T, W) = \sum_{i=1}^{n} \sum_{j=1}^{m} a_i b_j \text{Cov}(X_i, Y_j).
\]

**Proof.** We have by the definition of covariance (Definition 2.5.1) that

\[
\text{Cov}(T, W) = E[(T - \mu_T)(W - \mu_W)] \quad \text{where } E[T] = \mu_T \text{ is the mean of } T \text{ and } E[W] = \mu_W \text{ is the mean of } W
\]

\[
= E \left[ \left( \sum_{i=1}^{n} a_i X_i - \sum_{i=1}^{n} a_i E[X_i] \right) \left( \sum_{j=1}^{m} b_j Y_j - \sum_{j=1}^{m} b_j E[Y_j] \right) \right]
\]

\[
= E \left[ \left( \sum_{i=1}^{n} (a_i X_i - a_i E[X_i]) \right) \left( \sum_{j=1}^{m} (b_j Y_j - b_j E[Y_j]) \right) \right]
\]
Theorem 2.8.2 (continued 1)

Proof (continued).

\[
\text{Cov}(T, W) = E \left[ \sum_{i=1}^{n} \sum_{j=1}^{m} (a_i X_i - a_i E[X_i])(b_j Y_j - b_j E[Y_j]) \right]
\]

\[
= \sum_{i=1}^{n} \sum_{j=1}^{m} E[(a_i X_i - a_i E[X_i])(b_j Y_j - b_j E[Y_j])]
\]

since \(E\) is linear by Theorem 2.1.1

\[
= \sum_{i=1}^{n} \sum_{j=1}^{m} E[a_i b_j X_i Y_j - a_i b_j E[X_i] Y_j]
\]

\[-a_i b_j E[X_i] Y_j + a_i b_j E[X_i] E[Y_j])
\]

\[
= \sum_{i=1}^{n} \sum_{j=1}^{m} (a_i b_j E[X_i Y_j] - a_i b_j E[X_i] E[Y_j])
\]

\[-a_i b_j E[X_i] E[Y_j] + a_i b_j E[X_i] E[Y_j]) \text{ by Theorem 2.1.1}
\]
Theorem 2.8.2 (continued 2)

Proof (continued). . . .

\[ \text{Cov}(T, W) = \sum_{i=1}^{n} \sum_{j=1}^{m} a_i b_j (E[X_i Y_j] - 2E[X_i]E[Y_j] + E[X_i]E[Y_j]) \]

\[ = \sum_{i=1}^{n} \sum_{j=1}^{m} a_i b_j E[(X_i - E[X_i])(Y_j - E[Y_j])] \text{ by Theorem 2.1.1} \]

\[ = \sum_{i=1}^{n} \sum_{j=1}^{m} a_i b_j \text{Cov}(X_i, Y_j) \text{ by Definition 2.5.1}, \]

as claimed. □
Corollary 2.8.1

Corollary 2.8.1. Let \( X_1, X_2, \ldots, X_n \) be random variables and define \( T = \sum_{i=1}^{n} a_i X_i \). Provided \( E[X_i^2] < \infty \) for \( i = 1, 2, \ldots, n \), then

\[
\text{Var}(T) = \sum_{i=1}^{n} a_i^2 \text{Var}(X_i) + s \sum_{i < j} a_i a_j \text{Cov}(X_i, X_j).
\]

Proof. By the definition of variance (Definition 1.9.2) we have

\[
\text{Var}(T) = E[(T - E[T])^2] = E[(T - E[T])(T - E[T])]
\]

\[
= \text{Cov}(T, T) \text{ by Definition 2.5.1}
\]

\[
= \sum_{i=1}^{n} \sum_{j=1}^{m} a_i a_j \text{Cov}(X_i, X_i) + \sum_{i=1}^{n} \sum_{j=1, j \neq i}^{m} a_i a_j \text{Cov}(X_i, X_j)
\]

\[
= \sum_{i=1}^{n} a_i^2 \text{Var}(X_i) + 2 \sum_{i=1}^{n} \sum_{j=i+1}^{m} a_i a_j \text{Cov}(X_i, X_j)
\]

since \( \text{Var}(X_i) = \text{Cov}(X_i, X_i) \) and \( \text{Cov}(X_i, X_j) \geq \text{Cov}(X_j, X_i) \).
Corollary 2.8.1

**Corollary 2.8.1.** Let $X_1, X_2, \ldots, X_n$ be random variables and define $T = \sum_{i=1}^{n} a_i X_i$. Provided $E[X_i^2] < \infty$ for $i = 1, 2, \ldots, n$, then

$$\text{Var}(T) = \sum_{i=1}^{n} a_i^2 \text{Var}(X_i) + \sum_{i<j} a_i a_j \text{Cov}(X_i, X_j).$$

**Proof.** By the definition of variance (Definition 1.9.2) we have

$$\text{Var}(T) = E[(T - E[T])^2] = E[(T - E[T])(T - E[T])].$$

$$= \text{Cov}(T, T)$$ by Definition 2.5.1

$$= \sum_{i=1}^{n} \sum_{j=1}^{m} a_i a_j \text{Cov}(X_i, X_i) + \sum_{i=1}^{n} \sum_{j=1, j \neq i}^{m} a_i a_j \text{Cov}(X_i, X_j)$$

$$= \sum_{i=1}^{n} a_i^2 \text{Var}(X_i) + 2 \sum_{i=1}^{n} \sum_{j=i+1}^{m} a_i a_j \text{Cov}(X_i, X_j)$$

since $\text{Var}(X_i) = \text{Cov}(X_i, X_i)$ and $\text{Cov}(X_i, X_j) > \text{Cov}(X_j, X_i)$
Corollary 2.8.1 (continued)

**Corollary 2.8.1.** Let $X_1, X_2, \ldots, X_n$ be random variables and define $T = \sum_{i=1}^{n} a_i X_i$. Provided $E[X_i^2] < \infty$ for $i = 1, 2, \ldots, n$, then

$$\text{Var}(T) = \sum_{i=1}^{n} a_i^2 \text{Var}(X_i) + s \sum_{i<j} a_i a_j \text{Cov}(X_i, X_j).$$

**Proof (continued).**

$$\text{Var}(T) = \sum_{i=1}^{n} a_i^2 \text{Var}(X_i) + 2 \sum_{i=1}^{n} \sum_{j=i+1}^{m} a_i a_j \text{Cov}(X_i, X_j)$$

$$= \sum_{i=1}^{n} a_i^2 \text{Var}(X_i) + 2 \sum_{i<j} a_i a_j \text{Cov}(X_i, X_j),$$

as claimed.
Corollary 2.8.2

**Corollary 2.8.2.** Let \( X_1, X_2, \ldots, X_n \) be independent random variables and define \( T = \sum_{i=1}^{n} a_i X_i \). With \( \text{Var}(X_i) = \sigma_i^2 \) for \( i = 1, 2, \ldots, n \) we have

\[
\text{Var}(T) = \sum_{i=1}^{n} a_i^2 \sigma_i^2.
\]

**Proof.** Since \( X_i \) and \( X_j \) are independent for \( i \neq j \) then by Theorem 2.5.2 \( \text{Cov}(X_i, X_j) = 0 \) for \( i \neq j \). So by Corollary 2.8.1,

\[
\text{Var}(T) = \text{Cor}(T, T) = \sum_{i=1}^{n} a_i^2 \text{Var}(X_i) + 2 \sum_{i<j} a_i a_j \text{Cov}(X_i, X_j)
\]

\[
= \sum_{i=1}^{n} a_i \sigma_i^2 + 0 = \sum_{i=1}^{n} a_i \sigma_i^2,
\]

as claimed. \( \square \)
Corollary 2.8.2

**Corollary 2.8.2.** Let $X_1, X_2, \ldots, X_n$ be independent random variables and define $T = \sum_{i=1}^{n} a_i X_i$. With $\text{Var}(X_i) = \sigma_i^2$ for $i = 1, 2, \ldots, n$ we have $\text{Var}(T) = \sum_{i=1}^{n} a_i^2 \sigma_i^2$.

**Proof.** Since $X_i$ and $X_j$ are independent for $i \neq j$ then by Theorem 2.5.2 $\text{Cov}(X_i, X_j) = 0$ for $i \neq j$. So by Corollary 2.8.1,

$$
\text{Var}(T) = \text{Cor}(T, T) = \sum_{i=1}^{n} a_i^2 \text{Var}(X_i) + 2 \sum_{i<j} a_i a_j \text{Cov}(X_i, X_j)
$$

$$
= \sum_{i=1}^{n} a_i \sigma_i^2 + 0 = \sum_{i=1}^{n} a_i \sigma_i^2,
$$

as claimed.
Theorem 2.8.A

Theorem 2.8.A. Let $X_1, X_2, \ldots, X_n$ be independent and identically distributed random variables with common mean $\mu$ and variance $\sigma^2$. We have

$$E[\bar{X}] = \mu, \; \text{Var}(\bar{X}) = \frac{\sigma^2}{n}, \; S^2 = \frac{\sum_{i=1}^{n} X_i^2 - n\bar{X}^2}{n-1}, \text{ and } E[S^2] = \sigma^2.$$ 

Proof. We have $\bar{X} = \frac{\sum_{i=1}^{n} X_i}{n} = \sum_{i=1}^{n} \frac{X_i}{n}$ by definition, so by Theorem 2.8.1

$$E[\bar{X}] = \sum_{i=1}^{n} \frac{1}{n} E[X_i] = \sum_{i=1}^{n} \frac{\mu}{n} = \mu.$$
Theorem 2.8.A

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$$E[\bar{X}] = \mu, \quad \text{Var}(\bar{X}) = \frac{\sigma^2}{n}, \quad S^2 = \frac{\sum_{i=1}^{n} X_i^2 - n \bar{X}^2}{n-1}, \quad \text{and} \quad E[S^2] = \sigma^2.$$

Proof. We have $\bar{X} = \frac{\sum_{i=1}^{n} X_i}{n} = \frac{\sum_{i=1}^{n} X_i}{n}$ by definition, so by Theorem 2.8.1

$$E[\bar{X}] = \sum_{i=1}^{n} \frac{1}{n} E[X_i] = \sum_{i=1}^{n} \frac{\mu}{n} = \mu.$$

By Corollary 2.8.2, $\text{Var}(\bar{X}) = \sum_{i=1}^{n} \frac{1}{n^2} \sigma^2 = \frac{\sigma^2}{n}$. 
Theorem 2.8.A.

Let \( X_1, X_2, \ldots, X_n \) be independent and identically distributed random variables with common mean \( \mu \) and variance \( \sigma^2 \). We have

\[
E[\bar{X}] = \mu, \quad \text{Var}(\bar{X}) = \frac{\sigma^2}{n}, \quad S^2 = \frac{\sum_{i=1}^{n} X_i^2 - n\bar{X}^2}{n-1}, \quad \text{and} \quad E[S^2] = \sigma^2.
\]

Proof. We have

\[
\bar{X} = \frac{\sum_{i=1}^{n} X_i}{n} = \sum_{i=1}^{n} \frac{X_i}{n} \quad \text{by definition},
\]

so by Theorem 2.8.1

\[
E[\bar{X}] = \sum_{i=1}^{n} \frac{1}{n} E[X_i] = \sum_{i=1}^{n} \frac{\mu}{n} = \mu.
\]

By Corollary 2.8.2, \( \text{Var}(\bar{X}) = \sum_{i=1}^{n} \frac{1}{n^2} \sigma^2 = \frac{\sigma^2}{n} \). The proof that

\[
S^2 = \frac{\sum_{i=1}^{n} X_i^2 - n\bar{X}^2}{n-1}
\]

is to be given in Exercise 2.8.1.
Theorem 2.8.A

**Theorem 2.8.A.** Let $X_1, X_2, \ldots, X_n$ be independent and identically distributed random variables with common mean $\mu$ and variance $\sigma^2$. We have

$$E[\bar{X}] = \mu, \quad \text{Var}(\bar{X}) = \frac{\sigma^2}{n}, \quad S^2 = \frac{\sum_{i=1}^{n} X_i^2 - n\bar{X}^2}{n-1}, \quad \text{and} \quad E[S^2] = \sigma^2.$$  

**Proof.** We have $\bar{X} = \frac{\sum_{i=1}^{n} X_i}{n} = \sum_{i=1}^{n} \frac{X_i}{n}$ by definition, so by Theorem 2.8.1

$$E[\bar{X}] = \sum_{i=1}^{n} \frac{1}{n} E[X_i] = \sum_{i=1}^{n} \frac{\mu}{n} = \mu.$$  

By Corollary 2.8.2, $\text{Var}(\bar{X}) = \sum_{i=1}^{n} \frac{1}{n^2} \sigma^2 = \frac{\sigma^2}{n}$. The proof that

$$S^2 = \frac{\sum_{i=1}^{n} X_i^2 - n\bar{X}^2}{n-1}$$  

is to be given in Exercise 2.8.1.
Theorem 2.8.A (continued)

Proof (continued). Finally,

\[
E[S^2] = E \left[ \frac{\sum_{i=1}^{n} X_i^2 - n\bar{X}^2}{n-1} \right]
\]

\[
= \frac{\sum_{i=1}^{n} E[X_i^2] - nE[\bar{X}^2]}{n-1} \quad \text{by Theorem 2.1.1}
\]

\[
= \frac{n(\sigma^2 - \mu^2) - nE[\bar{X}^2]}{n-1} \quad \text{since } E[X^2] = \sigma^2 + \mu^2 \text{ by Note 1.9.A}
\]

\[
= \frac{n(\sigma^2 + \mu^2) - n(\sigma^2/n + \mu^2)}{n-1} \quad \text{since } E(\bar{X}^2) = \text{Var}(\bar{X})^2 + E(\bar{X})^2
\]

\[
= \frac{\sigma^2}{n} + \mu^2
\]

\[
= \frac{(n-1)\sigma^2}{n-1} = \sigma^2,
\]

as claimed.