Mathematical Statistics 1

Chapter 3. Some Special Distributions

3.2. The Poisson Distribution—Proofs of Theorems

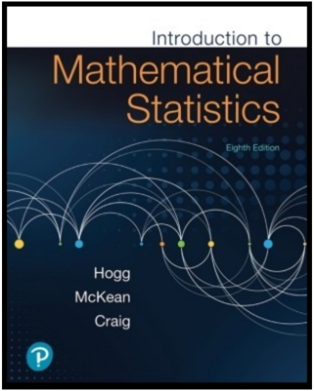

Table of contents

Theorem 3.2.1

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Proof. The moment generating function of X_i is $M_{X_i}(t) = e^{\lambda_i(e^t-1)}$. So by Theorem 2.6.1 the moment generating function of Y is

$$M_Y(t) = \prod_{i=1}^n M_{X_i}(t) = \prod_{i=1}^n e^{\lambda_i(e^t-1)} = \exp\left(\sum_{i=1}^n \lambda_i(e^t-1)\right).$$

By the uniqueness of moment generating functions (Theorem 1.9.2), we have that Y has a Poisson distribution with parameter $\sum_{i=1}^n \lambda_i$, as claimed □

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Exercise 3.2.13

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Solution. We have a Poisson distribution with parameter $\lambda = 3$ (articles per week), so the probability mass functions is

$$p(x) = \begin{cases} \frac{\lambda^x e^{-\lambda}}{x!} & \text{for } x \in \{0, 1, 2, \ldots\} \\ 0 & \text{otherwise} \end{cases} = \begin{cases} \frac{3^x e^{-3}}{x!} & \text{for } x \in \{0, 1, 2, \ldots\} \\ 0 & \text{otherwise.} \end{cases}$$

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We want the value of k such that the probability that not more then k articles are sold in a week is at least $1 - 0.01 = 0.99$ (so that the probability that the grocer runs out of articles is at most 0.01). So we want $\displaystyle\sum_{x=0}^{k} \frac{3^x e^{-3}}{x!} \geq 0.99$.

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We want the value of k such that the probability that not more then k articles are sold in a week is at least $1 - 0.01 = 0.99$ (so that the probability that the grocer runs out of articles is at most 0.01). So we want $\displaystyle\sum_{x=0}^{k} \frac{3^x e^{-3}}{x!} \geq 0.99$.

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Solution (continued). So we want $\sum_{x=0}^{k} \frac{3^x e^{-3}}{x!} \geq 0.99$. We appeal to software to solve this for the smallest value of k. Using the Poisson distribution applet of Matt Bognar on his University of Iowa webpage, we find the following values of k and associated probabilities:

k	$\mathbf{P(X \leq k)}$
5	0.91608
6	0.96649
7	0.98881
8	0.9962

So the grocer should buy $k = 8$ articles. □