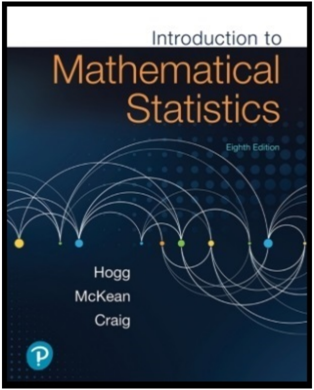
Mathematical Statistics 1

Chapter 3. Some Special Distributions

3.3. The Γ, χ^2, and β Distributions—Proofs of Theorems



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Theorem 3.3.1

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Proof. The hypothesized independence gives, by Theorem 2.6.1, that

$$\begin{aligned} M_Y(t) &= \prod_{i=1}^n M_i(t) \\ &= \prod_{i=1}^n \frac{1}{(1-\beta t)^{\alpha_i}} \text{ for } t < 1/\beta \text{ by Note 3.3.C} \\ &= (1-\beta t)^{-\sum_{i=1}^n \alpha_i} \text{ for } t < 1/\beta \end{aligned}$$

which is the moment generating function of a $\Gamma(\sum_{i=1}^n \alpha_i, \beta)$ distribution. By the uniqueness of the moment generating function (Theorem 1.9.2), the result follows. □

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Theorem 3.3.2

Theorem 3.3.2. Let X have a $\chi^2(r)$ distribution. If $k > -r/2$ then $E(X^k)$ exists and is

$$E(X^k) = \frac{2^k \Gamma(r/2+k)}{\Gamma(r/2)} \text{ if } k > -r/2.$$

Proof. We have

$$\begin{aligned} E(X^k) &= \int_{-\infty}^{\infty} x^k f(x)\,dx \\ &= \int_0^{\infty} x^k \frac{1}{\Gamma(r/2)2^{r/2}} x^{r/2-1} e^{-x/2}\,dx \\ &= \int_0^{\infty} \frac{1}{\Gamma(r/2)2^{r/2}} x^{r/2+l-1} e^{-x/2}\,dx \\ &\quad [\text{left } u = x/2 \text{ and } du = 1/2\,dx \text{ or } 2\,du = dx] \\ &= \int_0^{\infty} \frac{1}{\Gamma(r/2)2^{r/2-1}} 2^{r/2+k-1} u^{r/2+k-1} e^{-u}\,du \end{aligned}$$

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Theorem 3.3.2 (continued)

Theorem 3.3.2. Let X have a $\chi^2(r)$ distribution. If $k > -r/2$ then $E(X^k)$ exists and is

$$E(X^k) = \frac{2^k \Gamma(r/2 + k)}{\Gamma(r/2)} \text{ if } k > -r/2.$$

Proof. We have

$$\begin{aligned} E(X^k) &= \int_0^\infty \frac{1}{\Gamma(r/2) 2^{r/2-1}} 2^{r/2+k-1} u^{r/2+k-1} e^{-u}\, du \\ &= \frac{2^k}{\Gamma(r/2)} \int_0^\infty u^{(r/2+k)-1} e^{-u}\, du \\ &= \frac{2^k}{\Gamma(r/2)} \Gamma(r/2+k) \text{ for } r/2 + k > 0. \end{aligned}$$

So $E(X^k) = 2^k \Gamma(r/2+k)/\Gamma(r/2)$ for $k > -r/2$, as claimed. □

Corollary 3.3.1

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Proof. Since a $\chi^2(r_i)$-distribution is also a $\Gamma(r/2, 2)$ distribution, then by Theorem 3.3.1 the distribution of Y is

$$\Gamma\left(\sum_{i=1}^n (r_i/2), 2\right) = \Gamma\left(\frac{1}{2}\left(\sum_{i=1}^n r_i\right), 2\right),$$

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