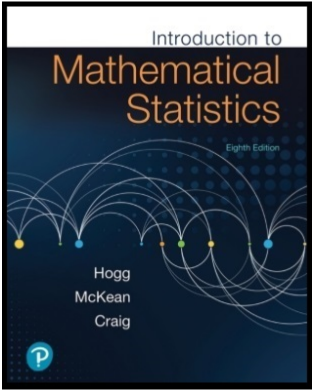
Mathematical Statistics 1

Chapter 3. Some Special Distributions

3.6. t- and F-Distributions—Proofs of Theorems



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Theorem 3.6.1

Theorem 3.6.1. Student's Theorem.
Let $X_1, X_2, \ldots, X_n$ be identical in distribution ("iid") random variables each having a normal distribution with mean μ and variance σ^2. Define the random variables

$$\overline{X} = \frac{1}{n}\sum_{i=1}^{n} X_i \text{ and } S^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \overline{X})^2.$$

Then

(a) $\overline{X}$ has a $N(\mu, \sigma^2/n$ distribution.

(b) $\overline{X}$ and S^2 are independent.

(c) $(n-1)S^2/\sigma^2$ has a $\chi^2(n-1)$ distribution.

(d) The random variable $T = \dfrac{\overline{X} - \mu}{S/\sqrt{n}}$ has a Student t-distribution with $n-1$ degrees of freedom.

Proof. **(a)** This is Corollary 3.4.1.

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Proof. (a) This is Corollary 3.4.1.

Theorem 3.6.1 (continued 1)

(b) $\overline{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$ and $S^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \overline{X})^2$ are independent.

Proof (continued). (b) Introduce vector $\mathbf{X} = (X_1, x_2, \ldots, X_n)'$. Since $X_1, X_2, \ldots, X_n$ are iid $N(\mu, \sigma^2)$ random variables, then by definition (see Definition 3.5.1), $\mathbf{X}$ has the multivariate normal distribution $N(\mu\mathbf{1}, \sigma^2\mathbf{1})$ where $\mathbf{1}$ is a column vector whose components are all 1. Let $\mathbf{v}' = (1/n)\mathbf{1}'$ (a row vector). Then $\mathbf{X} = \mathbf{v}'\mathbf{X}$ (notice that we effectively have a dot product here). Define the random vector $\mathbf{Y}$ as $\mathbf{Y} = (X_1 - \overline{X}, X_2 - \overline{X}, \ldots, X_n - \overline{X})'$. Consider

$$\mathbf{W} = \begin{bmatrix} \overline{X} \\ \mathbf{Y} \end{bmatrix} = \begin{bmatrix} \mathbf{v}' \\ \mathbf{I} - \mathbf{1}\mathbf{v}' \end{bmatrix} \mathbf{X}.$$

So $\mathbf{W}$ is a linear transformation of $\mathbf{X}$ of the form $\mathbf{W} = \mathbf{A}\mathbf{X} + \mathbf{b}$ where $\mathbf{A} = \begin{bmatrix} \mathbf{v}' \\ \mathbf{I} - \mathbf{1}\mathbf{v}' \end{bmatrix}$ and $\mathbf{b} = \mathbf{0}_{n+1}$ and hence by Theorem 3.5.2 $\mathbf{W}$ has mean $\mathbf{A}\mu + \mathbf{b}$ or...

Theorem 3.6.1 (continued 1)

(b) $\overline{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$ and $S^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \overline{X})^2$ are independent.

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Theorem 3.6.1 (continued 2)

(b) $\overline{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$ and $S^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \overline{X})^2$ are independent.

Proof (continued). ...

$$E[\mathbf{W}] = \begin{bmatrix} \mathbf{v}' \\ \mathbf{I} - \mathbf{1}\mathbf{v}' \end{bmatrix} \mu\mathbf{1} = \begin{bmatrix} \mu \\ \mu\mathbf{1} - \mu\mathbf{1}\mathbf{v}'\mathbf{1} \end{bmatrix}$$

$$= \begin{bmatrix} \mu \\ \mu(\mathbf{1} - \mathbf{1}\left(\frac{1}{n}\mathbf{1}'\right)\mathbf{1} \end{bmatrix} = \begin{bmatrix} \mu \\ \mu(\mathbf{1} - \mathbf{1}\left(\frac{1}{n}\right)(n) \end{bmatrix} = \begin{bmatrix} \mu \\ \mathbf{0}_n \end{bmatrix}$$

where $\mathbf{0}_n$ denotes a $n \times 1$ column vector of 0s. Also by Theorem 3.5.2, the covariance of $\mathbf{W}$ is $\mathbf{A}\boldsymbol{\Sigma}\mathbf{A}'$ where $\mathbf{A}$ is given above and $\boldsymbol{\Sigma} = \sigma^2\mathbf{I}$ (since $\mathbf{X}$ is $N(\boldsymbol{\mu}, \boldsymbol{\Sigma}) = N(\mu\mathbf{1}, \sigma^2\mathbf{1})$. Representing the covariance of $\mathbf{W}$ as $\boldsymbol{\Sigma}$ (as is standard notation) we have

$$\boldsymbol{\Sigma} = \begin{bmatrix} \mathbf{v}' \\ \mathbf{I} - \mathbf{1}\mathbf{v}' \end{bmatrix} \sigma^2\mathbf{I} \begin{bmatrix} \mathbf{v}' \\ \mathbf{I} - \mathbf{1}\mathbf{v}' \end{bmatrix}' \ldots$$

Theorem 3.6.1 (continued 3)

Proof (continued). ...

$$\begin{aligned}
\boldsymbol{\Sigma} &= \begin{bmatrix} \mathbf{v}' \\ \mathbf{I} - \mathbf{1}\mathbf{v}' \end{bmatrix} \sigma^2 \mathbf{I} \begin{bmatrix} \mathbf{v}' \\ \mathbf{I} - \mathbf{1}\mathbf{v}' \end{bmatrix}' = \sigma^2 \begin{bmatrix} \mathbf{v}' \\ \mathbf{I} - \mathbf{1}\mathbf{v}' \end{bmatrix} \begin{bmatrix} \mathbf{v}' \\ \mathbf{I} - \mathbf{1}\mathbf{v}' \end{bmatrix}' \\
&= \sigma^2 \begin{bmatrix} \mathbf{v}' \\ \mathbf{I} - \mathbf{1}\mathbf{v}' \end{bmatrix} [\mathbf{v} \quad \mathbf{I} - \mathbf{v}\mathbf{1}'] \\
&= \sigma^2 \begin{bmatrix} n(1/n^2) & \mathbf{v}' - \mathbf{v}'\mathbf{1}' \\ \mathbf{v} - \mathbf{1}\mathbf{v}'\mathbf{v} & (\mathbf{I} - \mathbf{v}\mathbf{1}' - \mathbf{1}\mathbf{v}' + \mathbf{1}\mathbf{v}'\mathbf{v}\mathbf{1}') \end{bmatrix} \\
&= \sigma^2 \begin{bmatrix} 1/n & \mathbf{0}_n \\ \mathbf{0}_n & \mathbf{I} - \mathbf{1}\mathbf{v}' \end{bmatrix} \text{ since}
\end{aligned}$$

$\mathbf{v} - \mathbf{1}\mathbf{v}'\mathbf{v} = \mathbf{v} = \mathbf{1}\left(\frac{1}{n}\mathbf{1}'\right)\mathbf{v} = \mathbf{v} - \frac{1}{n}\mathbf{1}\mathbf{1}'\mathbf{v} = \mathbf{v} - \frac{1}{n}(n)\mathbf{v} = \mathbf{v} - \mathbf{v} = \mathbf{0}_n$ and

$$\begin{aligned}
\mathbf{I} - \mathbf{v}\mathbf{1}' - \mathbf{1}\mathbf{v}' + \mathbf{1}\mathbf{v}'\mathbf{v}\mathbf{1}' &= \mathbf{I} - \mathbf{v}\mathbf{1}' - \mathbf{1}\mathbf{v}' + \mathbf{1}\left(\frac{1}{n}\mathbf{1}'\right)\mathbf{v}\mathbf{1} \\
&= \mathbf{I} - \mathbf{v}\mathbf{1}' - \mathbf{1}\mathbf{v}' + \frac{1}{n}(n)\mathbf{v}\mathbf{1}' = \mathbf{I} - \mathbf{1}\mathbf{v}'.
\end{aligned}$$

Theorem 3.6.1 (continued 4)

(b) $\overline{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$ and $S^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \overline{X})^2$ are independent.

(c) $(n-1)S^2/\sigma^2$ has a $\chi^2(n-1)$ distribution.

Proof (continued). By Theorem 3.5.3, since the covariances are 0, then $\overline{X}$ is independent of $\mathbf{Y}$. Since

$$S^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \overline{X})^2 = (n-1)^{-1}\mathbf{Y}'\mathbf{Y},$$

then $\overline{X}$ is also independent of S^2, and (b) follows.

(c) To explore the distribution of $(n-1)S^2/\sigma^2$, consider random variable $V = \sum_{i=1}^{n}\left(\frac{X_i - \mu}{\sigma}\right)^2$. For each $i = 1, 2, \ldots, n$, $(X_i - \mu)^2/\sigma^2$ has a $\chi^2(1)$ distribution. Since the X_i are independent by hypothesis, then the $(X_i - \mu)^2/\sigma^2$ are independent and so by Corollary 3.3.1, V is a $\chi^2(n)$ random variable.

Theorem 3.6.1 (continued 4)

(b) $\overline{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$ and $S^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \overline{X})^2$ are independent.

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Proof (continued). By Theorem 3.5.3, since the covariances are 0, then $\overline{X}$ is independent of $\mathbf{Y}$. Since

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(c) To explore the distribution of $(n-1)S^2/\sigma^2$, consider random variable $V = \sum_{i=1}^{n}\left(\frac{X_i - \mu}{\sigma}\right)^2$. For each $i = 1, 2, \ldots, n$, $(X_i - \mu)^2/\sigma^2$ has a $\chi^2(1)$ distribution. Since the X_i are independent by hypothesis, then the $(X_i - \mu)^2/\sigma^2$ are independent and so by Corollary 3.3.1, V is a $\chi^2(n)$ random variable.

Theorem 3.6.1 (continued 5)

Proof (continued). We also have

$$\begin{aligned} V &= \sum_{i=1}^{n} \left(\frac{X_i - \mu}{\sigma} \right)^2 = \sum_{i=1}^{n} \left(\frac{(X_i - \overline{X}) + (\overline{X} - \mu)}{\sigma} \right)^2 \\ &= \left(\sum_{i=1}^{n} \left(\frac{X_i - \overline{X}}{\sigma} \right)^2 \right) + \left(\frac{\overline{X} - \mu}{\sigma/\sqrt{n}} \right)^2 \\ &= \frac{n-1}{\sigma^2} \left(\frac{1}{n-1} \sum_{i=1}^{n} (X_i - \overline{X})^2 \right) + \left(\frac{\overline{X} - \mu}{\sigma/\sqrt{n}} \right)^2 \\ &= \frac{(n-1)S^2}{\sigma^2} + \left(\frac{\overline{X} - \mu}{\sigma/\sqrt{n}} \right)^2 . \qquad (*) \end{aligned}$$

By part (b), $\overline{X}$ and S^2 are independent, so $\dfrac{(n-1)S^2}{\sigma^2}$ and $\left(\dfrac{\overline{X} - \mu}{\sigma/\sqrt{n}} \right)^2$ are independent.

Theorem 3.6.1 (continued 6)

Proof (continued). Now $\left(\frac{\overline{X}-\mu}{\sigma/\sqrt{n}}\right)^2$ is the square of a standard normal random variable by Corollary 3.4.1 and so by Theorem 3.4.1 $\left(\frac{\overline{X}-\mu}{\sigma/\sqrt{n}}\right)^2$ has a $\chi^2(1)$ distribution. Since V is a $\chi^2(n)$ random variable then the moment generating function of V is $(1-2t)^{-n/2}$ by Example 3.3.4. Since $\left(\frac{\overline{X}-\mu}{\sigma/\sqrt{n}}\right)^2$ has a $\chi^2(1)$ distribution, then the moment generating function is $(1-2t)^{-1/2}$ by Example 3.3.4. Now for random variables X and Y, the moment generating function of $X+Y$ is

$$E(e^{t(X+Y)}) = E(e^{tX}e^{tY}) = E(e^{tX})E(e^{tY})$$

by Theorem 2.4.4.

Theorem 3.6.1 (continued 7)

(c) $(n-1)S^2/\sigma^2$ has a $\chi^2(n-1)$ distribution.

Proof (continued). So from $(*)$ we have the moment generating functions

$$(1-2t)^{-n/2} = E[\exp(t(n-1(S^2/\sigma^2)](1-2t)^{-1/2},$$

and hence the moment generating function of $(n-1)S^2/\sigma^2$ is

$$E[\exp(t(n-1)S^2/\sigma^2)] = (1-2t)^{-(n-1)/2}.$$

This is the moment generating function of a $\chi^2(n-1)$ distribution by Example 3.3.4. By the uniqueness of the moment generating function (Theorem 1.9.2), we have that $(n-1)S^2/\sigma^2$ has a $\chi^2(n-1)$ distribution, and (c) follows.

Theorem 3.6.1 (continued 8)

(d) The random variable $T = \dfrac{\overline{X} - \mu}{S/\sqrt{n}}$ has a Student t-distribution with $n-1$ degrees of freedom.

Proof (continued). (d) We have

$$T = \frac{\overline{X} - \mu}{S/\sqrt{n}} = \frac{(\overline{X} - \mu)/(\sigma/\sqrt{n})}{\sqrt{S^2/\sigma^2}} = \frac{(\overline{X} - \mu)/(\sigma/\sqrt{n})}{\sqrt{(n-1)S^2/(\sigma^2(n-1))}}.$$

So $T = \dfrac{W}{\sqrt{V/r}}$ where $W = (\overline{X} - \mu)/(\sigma/\sqrt{n})$ has a $N(0,1)$ distribution (by Corollary 3.4.1), $V = (n-1)S^2/\sigma^2$ has a $\chi^2(n-1)$ distribution by part (c), and $r = n-1$. Also, W and V are independent by part (b). Therefore, by Note 3.6.A, T has a t-distribution with r degrees of freedom, and (d) follows. □