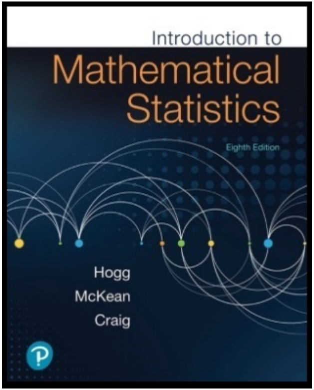
Mathematical Statistics 1

Chapter 4. Some Elementary Statistical Inferences

4.2. Confidence Intervals—Proofs of Theorems



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Exercise 4.2.1

Exercise 4.2.1. Let the observed value of the mean $\overline{X}$ and the sample variance of a ransom sample of size 20 from a distribution that is $N(\mu, \sigma^2)$ be 81.2 and 25.6, respectively. First, the 90%, 95%, and 99% confidence intervals for μ. Not how the lengths of the confidence intervals increase as the confidence increases.

Solution. We are given $n = 20$, $\overline{x} = 81.2$, and $s = 26.5$. We are interested in $\alpha \in \{0.10, 0.05, 0.01\}$, so we need $t_{\alpha/2,19}$ for these values. We find (from software or a table) that

$$t_{0.05,19} = 1.729,\ t_{0.025,19} = 2.093,\ \text{and } t_{0.005,19} = 2.861,$$

so that the confidence intervals are

$$90\%:\ \ (81.2 - 1.729(26.5)/\sqrt{2}, 81.2 + 1.729(26.5)/\sqrt{2}) = (70.95, 91.45),$$
$$95\%:\ \ (81.2 - 2.093(26.5)/\sqrt{2}, 81.2 + 2.093(26.5)/\sqrt{2}) = (68.80, 93.60),$$
$$99\%:\ \ (81.2 - 2.861(26.5)/\sqrt{2}, 81.2 + 2.861(26.5)/\sqrt{2}) = (64.25, 98.15).$$

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\end{aligned}$$

Exercise 4.2.1 (continued)

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Solution (continued). Since the confidence intervals are

$$\begin{aligned} 90\% &: \quad (70.95, 91.45), \\ 95\% &: \quad (68.80, 93.60), \\ 99\% &: \quad (64.25, 98.15), \end{aligned}$$

we observe that the greater the confidence, the larger the interval (as we should expect). $\square$

Exercise 4.2.A

Exercise 4.2.A. (From William Navadi's *Statistics for Engineers and Scientists*, 3rd Edition, McGraw-Hill [2011]). In a simple random sample of 70 automobiles registered in a certain state, 28 or them were found to have emission levels that exceed a state standard. Find 95% and 98% confidence intervals for the proportion of automobiles in the state whose emission levels exceed the standard.

Solution. Let X be the Bernoulli random variable X where $X = 1$ when automobile's emission level exceeds the state standard (a "success") and $X = 0$ otherwise. For the sample, we have $n = 70$ and $\overline{X} = 28/70 = 0.40 = \hat{p}$. For $\alpha = 0.05$ we have $z_{\alpha/2} = z_{0.025} = 1.96$, and for $\alpha 0.02$ we have $Z_{\alpha/2} = z_{0.01} = 2.33$. So the 95% confidence interval for the proportion p of automobile in the state whose emission levels exceed the standard is ...

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Exercise 4.2.A (continued)

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Solution (continued). ...

$$\left(\hat{p} - z_{0.025}\sqrt{\hat{p}(1-\hat{p})/n}, \hat{p} + z_{0.025}\sqrt{\hat{p}(1-\hat{p})/n}\right)$$

$$= \left(0.40 - 1.96\sqrt{0.40(1-0.40)/70}, 0.40 + 1.96\sqrt{0.40(1-0.40)/70}\right)$$

$$= (0.29, 0.51)$$

and similarly the 98% confidence interval is

$$\left(0.40 - 2.33\sqrt{0.40(1-0.40)/70}, 0.40 + 2.33\sqrt{0.40(1-0.40)/70}\right)$$

$$= (0.26, 0.54). \qquad \square$$