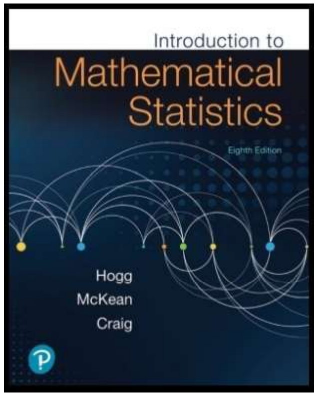
Mathematical Statistics 1

Chapter 4. Some Elementary Statistical Inferences

4.3. Confidence Intervals for Parameters of Discrete Distributions—Proofs of Theorems



Theorem 4.2.A

Theorem 4.2.A. Consider a sample $X_1, X_2, \ldots, X_n$ on a discrete random variable X with probability mass function $p(x;\theta)$, where $\theta \in \Omega$ and Ω is an interval of real numbers. Let $T = T(X_1, X_2, \ldots, X_n)$ be an estimator of θ where the cumulative distribution function of T is $F_T(t;\theta)$. Suppose that $F(t;\theta)$ is a nonincreasing and continuous function of θ for every t in the support of T. For a given realization $x_1, x_2, \ldots, x_n$ of the sample, let t be the realized value of the statistic T (so $t = T(x_1, x_2, \ldots, x_n)$). Let $\alpha_1 > 0$ and $\alpha_2 > 0$ be given such that $\alpha = \alpha_1 + \alpha_2 < 0.50$. Let $\underline{\theta}$ and $\overline{\theta}$ be the solutions of the equations

$$F_T(t-;\underline{\theta}) = 1 - \alpha_2 \text{ and } F_T(t;\overline{\theta}) = \alpha_1,$$

where $T-$ is the statistic whose support lags by one value of T's support. The interval $(\underline{\theta}, \overline{\theta})$ is a *confidence interval* for θ with confidence coefficient of at least $1 - \alpha$.

Theorem 4.2.A (continued)

Proof ("brief sketch"). The cumulative distribution function of T is $F_T(t;\theta)$, which is a nonincreasing function of θ for a given t. Define

$$\overline{\theta} = \sup\{\theta \mid F_T(T;\theta) \geq \alpha_1\} \text{ and } \underline{\theta} = \inf\{\theta \mid F_T(T-;\theta) \leq 1 - \alpha_2\}.$$

Since $F_T(t;\theta)$ is nonincreasing, if $\theta > \overline{\theta}$ then $F_T(T;\theta) \geq \alpha_1$, and if $\theta < \underline{\theta}$ then $F_T(T-;\theta) \leq 1 - \alpha_2$. So,

$$\begin{aligned} P(\underline{\theta} < \theta < \overline{\theta}) &= 1 - P(\theta \leq \underline{\theta} \text{ or } \theta \geq \overline{\theta}) = 1 - P(\{\theta \leq \underline{\theta}\} \cup \{\theta \geq \overline{\theta}\}) \\ &= 1 - P(\{\theta \leq \underline{\theta}\}) - P(\{\theta \geq \overline{\theta}\}) \\ &= 1 - P(\{\theta < \underline{\theta}\}) - P(\{\theta > \overline{\theta}\}) \text{ (this needs justification)} \\ &\geq 1 - P(F_T(T-;\theta) \geq 1 - \alpha_2) - P(F_T(T;\theta) \leq \alpha_1) \\ &\geq 1 - \alpha_2 - \alpha_1 = 1 - (\alpha_1 + \alpha_2) = 1 - \alpha. \end{aligned}$$

□