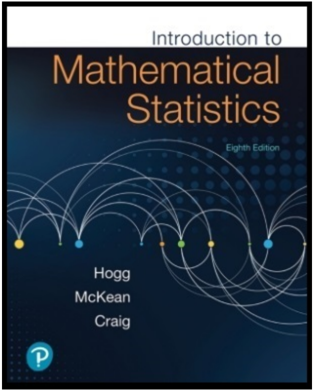
Mathematical Statistics 1

Chapter 5. Consistency and Limiting Distributions

5.3. Central Limit Theorem—Proofs of Theorems



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Since $m(t)$ is the moment generating function for $X - \mu$, then by Note 1.9.B, $m(0) = e^{-\mu(0)}M(0) = E(e^0) = 1$, $m'(0) = E(X - \mu) = 0$ (since $m'(0)$ is the mean of the random variable), and

$$m''(0) = E((X-\mu)^2) = E(X^2) - 2\mu E(X) + E(\mu^2)$$
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Theorem 5.3.1 (continued 1)

Partial Proof (continued). By "Taylor's Theorem or the Mean Value Theorem" (stated in Section 5.2) there exists ξ between 0 and t such that

$$m(t) = m(0) + m'(0)t + \frac{m''(\xi)}{2}t^2 = 1 + \frac{m''(\xi)}{2}t^2$$
$$= 1 + \frac{\sigma^2 t^2}{2}\frac{m''(\xi) - \sigma^2}{2}t^2. \quad (5.3.1)$$

Next, the moment generating function $M(n;t)$ of $Y_n = \left(\sum_{i=1}^n X_i - n\mu\right)/(\sqrt{n}\sigma)$ is

$$\begin{aligned} M(t;n) &= E\left(\exp\left(t\frac{\sum_{i=1}^n X_i - n\mu}{\sigma\sqrt{n}}\right)\right) \\ &= E\left(\exp\left(t\frac{X_1-\mu}{\sigma\sqrt{n}}\right)\exp\left(t\frac{X_2-\mu}{\sigma\sqrt{n}}\right)\cdots\exp\left(t\frac{X_n-\mu}{\sigma\sqrt{n}}\right)\right) \end{aligned}$$
$$= E\left(\exp\left(t\frac{X_1-\mu}{\sigma\sqrt{n}}\right)\right)E\left(\exp\left(t\frac{X_2-\mu}{\sigma\sqrt{n}}\right)\right)\cdots E\left(\exp\left(t\frac{X_1-\mu}{\sigma\sqrt{n}}\right)\right)$$

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Theorem 5.3.1 (continued 2)

Partial Proof (continued).

$$M(t;n) = E\left(\exp\left(t\frac{X_1-\mu}{\sigma\sqrt{n}}\right)\right) E\left(\exp\left(t\frac{X_2-\mu}{\sigma\sqrt{n}}\right)\right) \cdots E\left(\exp\left(t\frac{X_1-\mu}{\sigma\sqrt{n}}\right)\right)$$

by Theorem 2.4.4, since $X_1, X_2, \ldots, X_n$ are independent

$$= \left(E\left(\exp\left(t\frac{X-\mu}{\sigma\sqrt{n}}\right)\right)\right)^n$$ since X_i for $i = 1, 2, \ldots, n$ is a sample from the distribution of X

$$= \left(m\left(\frac{t}{\sigma\sqrt{n}}\right)\right)^n$$ for $-hM\frac{t}{\sigma\sqrt{n}} < h$ since $E(t(X-\mu)) = m(t)$ for $-h < t < h$.

From (5.3.1) we have (replacing t with $t/(\sigma\sqrt{n})$), ...

Theorem 5.3.1 (continued 3)

Partial Proof (continued). From (5.3.1) we have (replacing t with $t/(\sigma\sqrt{n})$),

$$m\left(\frac{t}{\sigma\sqrt{n}}\right) = 1 + \frac{\sigma^2}{2}\left(\frac{t}{\sigma\sqrt{n}}\right)^2 + \frac{m''(\xi)-\sigma^2}{2}\left(\frac{t}{\sigma\sqrt{n}}\right)^2$$

$$= 1 + \frac{t^2}{2n} + \frac{(m''(\xi)-\sigma^2)t^2}{2n\sigma^2}$$

for some ξ between 0 and $t/(\sigma\sqrt{n})$ with $-h\sigma\sqrt{n} < t < h\sigma\sqrt{n}$, This gives

$$M(t;n) = \left(1 + fract^2 2n + \frac{(m''(\xi)-\sigma^2)t^2}{2n\sigma^2}\right)^n.$$

Since $m''(t)$ is continuous at $t=0$ and since $\xi \to 0$ as $n \to \infty$ (because ξ is between 0 and $t/(\sigma\sqrt{n})$), then

$$\lim_{n\to\infty}(m''(\xi)-\sigma^2) = m''(0)-\sigma^2 = \sigma^2-\sigma^2 = 0.$$

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Partial Proof (continued). By Theorem 5.2.B we then have

$$\lim_{n\to\infty} M(t;n) = \lim_{n\to\infty} \left(1 + \frac{t^2}{2n} + \frac{(m''(\xi) - \sigma^2)t^2}{2n\sigma^2}\right)^n = e^{t^2/2} \text{ for } t \in \mathbb{R}.$$

This is the moment generating function of $N(0,1)$ by Note 3.4.A. So by Theorem 5.2.10,

$$Y_n = \frac{\sum_{i=1}^n X_i - n\mu}{\sigma\sqrt{n}} \xrightarrow{D} N(0,1),$$

as claimed. □